



University of Crete

Department of Mathematics and Applied Mathematics

April 29, 2026

Achilles Tertikas

Weak Partial Differential Equations (Homework 3)

Exercise 1 Let Ω be a bounded domain in $\mathbf{R}^3$ and $u \in \mathbf{H}_0^1(\Omega)$ is a weak solution of

$$\begin{aligned} -\Delta u(x) &= u^2(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Establish that $u \in \mathbf{C}^2(\Omega)$.

Exercise 2 (Neumann Problem) Let Ω be a bounded domain in $\mathbf{R}^n$ with $\mathbf{C}^1$ boundary and $f \in L^2(\Omega)$. Then $u \in \mathbf{H}^1(\Omega)$ is a weak solution of

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in \Omega \\ \frac{\partial u(x)}{\partial \nu} &= 0, \quad x \in \partial\Omega, \end{aligned}$$

if it satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in \mathbf{H}^1(\Omega).$$

If the previous problem has a $\mathbf{H}^1(\Omega)$ weak solution, then the following should hold true

$$\int_{\Omega} f(x) \, dx = 0. \quad (\star)$$

If f satisfies $(\star)$, then establish the existence of a weak solution to the Neumann problem.

Suggestion: Consider suitably the subspace

$$\mathcal{A} = \{u \in \mathbf{H}^1(\Omega), \int_{\Omega} u(x) \, dx = 0.\}$$

Exercise 3 Let Ω be a bounded domain in $\mathbf{R}^n$ with $\mathbf{C}^1$ boundary and $f \in L^2(\Omega)$. Define $u \in \mathbf{H}^1(\Omega)$ is a weak solution of

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in \Omega, \\ u(x) + \frac{\partial u(x)}{\partial \nu} &= 0, \quad x \in \partial\Omega, \end{aligned}$$

if it satisfies

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx + \int_{\partial\Omega} u(x) \phi(x) \, dS = \int_{\Omega} f \phi \, dx, \quad \forall \phi \in \mathbf{H}^1(\Omega).$$

Establish the existence of a weak solution.

Suggestion: : Establish that the relation

$$\|u\|_{\mathbf{H}} = \left(\int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial\Omega} |u|^2 \, dS \right)^{1/2}$$

defines an equivalent to the usual $\mathbf{H}^1(\Omega)$ norm.

You may assume that the following trace inequality holds true

$$\left(\int_{\partial\Omega} |u|^{\frac{2(n-1)}{n-2}} \, dS \right)^{\frac{n-2}{n-1}} \leq c \left(\int_{\Omega} |\nabla u|^2 + |u|^2 \, dx \right), \quad \forall u \in \mathbf{H}^1(\Omega),$$

and that we have the compactness

$$\mathbf{H}^1(\Omega) \hookrightarrow L^p(\partial\Omega) \text{ if } 1 \leq p < \frac{2(n-1)}{n-2}.$$

Exercise 4 Let Ω be a bounded domain in $\mathbf{R}^n$ with $\mathbf{C}^1$ boundary. Suppose in addition that the boundary $\partial\Omega$ consists of two $\mathbf{C}^1$ closed sets Γ_1, Γ_2 , with

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \text{ with } H^{n-1}(\Gamma_1) > 0, \quad H^{n-1}(\Gamma_2) > 0.$$

Define properly weak solution to the Dirichlet-Neumann problem:

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in \Omega \\ u(x) &= 0, \quad x \in \Gamma_1, \\ \frac{\partial u(x)}{\partial \nu} &= 0, \quad x \in \Gamma_2. \end{aligned}$$

Establish the existence of a $\mathbf{H}^1(\Omega)$ weak solution.

Exercise 5 (Helmholtz decomposition) Let Ω be a bounded domain in $\mathbf{R}^n$ and the vector field $\mathbf{b} \in (\mathbf{L}^2(\Omega))^n$. Establish the existence of $u \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{v} \in (\mathbf{L}^2(\Omega))^n$ such that

$$\mathbf{b}(x) = \nabla u(x) + \mathbf{v}(x)$$

with

$$\nabla \cdot \mathbf{v} = 0 \text{ weakly in } \Omega \text{ and } \int_{\Omega} \nabla u(x) \cdot \mathbf{v}(x) \, dx = 0.$$

Is the decomposition unique?