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STABILITY AND CONVERGENCE OF A CLASS OF FINITE ELEMENT SCHEMES FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS*

CHRISTOS ARVANITIS†, CHARALAMBOS MAKRIDAKIS‡, AND ATHANASIOS E. TZAVARAS§

Abstract. We propose a class of finite element schemes for systems of hyperbolic conservation laws that are based on finite element discretizations of appropriate relaxation models. We consider both semidiscrete and fully discrete finite element schemes and show that the schemes are stable and, when the compensated compactness theory is applicable, do converge to a weak solution of the hyperbolic system. The schemes use piecewise polynomials of arbitrary degree and their consistency error is of high order. We also prove that the rate of convergence of the relaxation system to a smooth solution of the conservation laws is of order $O(\varepsilon)$.

Key words. stability and convergence, finite element schemes, hyperbolic conservation laws, adaptive schemes

AMS subject classifications. 65M60, 65M12, 65M15, 35L65

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1. Introduction.

The problem of numerical approximation of nonlinear hyperbolic systems of conservation laws,

$$
\begin{aligned}
&\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = 0, \quad x \in \mathbb{R}^d, \ \ u = u(x,t) \in \mathbb{R}^n, \quad t > 0\,, \\
&u(\cdot, 0) = u_0(\cdot),
\end{aligned} \tag{1.1}
$$

is a challenging area testing the performance of various numerical methods. Such methods need to resolve accurately the shock regions and at the same time approximate with high accuracy the smooth parts of the solution.

It is a widely held belief that to achieve this goal one has to impose extraneous stabilization mechanisms, such as shock capturing terms or limiters (depending on the parameters of the problem, on the order of the method, on the particular form of the system, etc.). This approach seems to hold for those finite element or high-order finite volume methods previously developed [21, 10, 19, 11]. We refer to [11] for a comprehensive review of the current state of the art on the high-order finite difference, finite volume, and finite element methods for hyperbolic conservation laws; see also [17, 26].

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†Department of Applied Mathematics, University of Crete, 71409 Heraklion, Crete, Greece, and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Crete, Greece (arvas@tem.uoc.gr).

‡Department of Applied Mathematics, University of Crete, 71409 Heraklion, Crete, Greece, and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Crete, Greece (makr@math.uoc.gr, http://www.tem.uoc.gr/~makr).

§Department of Mathematics, University of Wisconsin, Madison, WI 53706, and Institute of Applied and Computational Mathematics, FORTH, 71110 Heraklion, Crete, Greece (tzavaras@math.wisc.edu, http://www.math.wisc.edu/~tzavaras).

Our motivation is to consider schemes designed to be used in conjunction with appropriate mesh refinement. It is conceivable that successful adaptive schemes may not need to be stabilized by using extra stabilization operators (such as limiters or shock capturing terms) accounting in turn for stabilization by the natural diffusion or relaxation mechanisms of the problem plus an appropriate mesh selection. A successful application of this idea requires one to have at hand a stable, robust, and flexible method. Indeed, toward this goal finite elements are a natural choice, since the development of supportive structures (finite element spaces of any order, flexibility in mesh construction, etc.) in adaptive finite element literature and software implementation is at a remarkable level.

In this article we propose a class of finite element methods based on relaxation models and address stability and convergence issues. For these relaxation finite element schemes the stabilization mechanisms are the regularization by wave operators (coming from the relaxation model) and appropriate mesh refinement in the shock areas. Our adaptive finite element schemes are of the general type introduced in [4] and further developed in [2, 3]. There, alternative methods and mesh refinement strategies are extensively tested computationally. Preliminary results indicate that the adaptive relaxation finite element schemes are a robust and reliable alternative for shock computations.

1.1. Relaxation finite element approximations. Relaxation models that approximate (1.1) are the basis of our schemes. In particular, the model suggested in [20],

$$
\begin{aligned}
&\partial_t\, u \;+\sum_{j=1}^{d}\partial_{x_j} v_j = \;0,\\
&\partial_t\, v_i \;+ A_i\,\partial_{x_i}\, u \;=\; -\frac{1}{\varepsilon}\big(v_i - F_i(u)\big), \quad i = 1,\dots,d,
\end{aligned}
\tag{1.2}
$$

corresponds to the regularization of (1.1) by a wave operator of order ε. Here A_i are symmetric, positive definite matrices with constant coefficients that are selected to satisfy certain stability conditions, *the subcharacteristic conditions*; see [20, 43] and the next sections. This relaxation model induces a regularization mechanism with *finite speed of propagation* that results in a partial differential equation with linear principal part. In return, the number of unknowns is increased. Nevertheless, in schemes based on the discretization of (1.2) the extra cost is compensated for by the simplicity and the natural implicit-explicit discretization that this model admits. The relaxation finite element schemes are based on the direct finite element approximation of (1.2).

Let $\mathcal{T}_h = \{K\}$ be a decomposition of $\mathbb{R}^d$ into elements with the usual properties [7]. We use the notation $h_K = \operatorname{diam}(K)$, $h = \sup_{K\in\mathcal{T}_h} h_K < 1$, and $\underline{h} = \min_{K\in\mathcal{T}_h} h_K$. The standard conforming finite element space S_k is defined by

$$
S_k = \{\phi \in C^0\,(\mathbb{R}^d)^n \;:\; \phi\,|_K \in \mathbb{P}_k,\; K\in\mathcal{T}_h,\;\; \phi\,|_{\Omega^C} \equiv\; 0\}. \tag{1.3}
$$

Here we assume that the initial values have compact support and thus, for all $t\in[0,T]$, our solution will vanish outside some compact set $\Omega\subset\mathbb{R}^n$. Clearly, $S_k \subset H_0^1(\Omega)$; see [7] for the approximation properties of S_k into Sobolev spaces. Further, we introduce a finite element space consisting of piecewise discontinuous polynomials:

$$
V_{k-1} = \{\psi \in L^2(\mathbb{R}^d)^n \;:\; \psi\,|_K \in \mathbb{P}_{k-1},\; K\in\mathcal{T}_h,\;\; \psi\,|_{\Omega^C} \equiv\; 0\}. \tag{1.4}
$$

By construction $\partial_{x_i}\phi \in V_{k-1}$ for all $\phi\in S_k$.

The schemes under consideration are obtained by a direct discretization (without adding additional diffusion terms) of (1.2). The approximation of u is sought in the space S_k and the approximations of the relaxation variables v_i in V_{k-1}; that is, find $(u_h, v_{h,1}, \ldots, v_{h,d}) : (0,T] \to S_k \times (V_{k-1})^d$ such that

(1.5)
$$\begin{aligned}(\partial_t\, u_h, \phi) \; - \sum_{j=1}^{d} (v_{h,j}, \partial_{x_j}\, \phi) \; &= \; 0 \quad \forall\, \phi \in S_k,\\ (\partial_t\, v_{h,i}, \psi) \; + \; (A_i\, \partial_{x_i}\, u_h, \psi) \; &= \; -\frac{1}{\varepsilon}\big(v_{h,i} - F_i(u_h), \psi\big) \quad \forall\, \psi \in V_{k-1}, i = 1, \ldots, d,\end{aligned}$$

with initial conditions $u_h(0) = \Pi_S u_0$ and $v_{h,i}(0) = \Pi_V F_i(u_0)$, where Π_S and Π_V are nodal interpolants on S_k and V_{k-1}, respectively. We note that (1.5) is a semidiscrete scheme since we have discretized only the spatial variable, in the sense that for any fixed $t \in [0,T]$, $u_h(\cdot, t) \in S_k$. In section 2 we show that if u_h solves (1.5), then it satisfies

$$\begin{aligned}(\partial_t\, u_h, \phi) + \sum_{i=1}^{d} \big(\partial_{x_i} F_i(u_h), \phi\big)&\\ + \; \varepsilon \left((\partial_{tt}\, u_h, \phi) + \sum_{i=1}^{d} (A_i\, \partial_{x_i}\, u_h, \partial_{x_i}\, \phi) \right) &= 0 \quad \forall \phi \in S_k.\end{aligned} \tag{1.6}$$

In the stability analysis we work with (1.6) but note that (1.5) is better suited to explicit-implicit one-step discretizations in time. Time discretizations based on (1.6) are also possible; see section 3.

The method is comparable, in terms of computational performance, with the fully conforming discretization of the relaxation model considered in [4]: find $(u_h, v_{h,1}, \ldots, v_{h,d}) : (0,T] \to (S_k)^{d+1}$ such that

$$\begin{aligned}(\partial_t\, u_h, \phi) \; - \sum_{j=1}^{d} (v_{h,j}, \partial_{x_j}\, \phi) \; &= \; 0 \quad \forall\, \phi \in S_k,\\ (\partial_t\, v_{h,i}, \psi) \; + \; (A_i\, \partial_{x_i}\, u_h, \psi) \; &= \; -\frac{1}{\varepsilon}\big(v_{h,i} - F_i(u_h), \psi\big) \quad \forall\, \psi \in S_k, i = 1, \ldots, d.\end{aligned} \tag{1.7}$$

The corresponding one field equation to (1.7) takes the form

$$\begin{aligned}(\partial_t\, u_h, \phi) + \sum_{i=1}^{d} \big(\partial_{x_i} P\, F_i(u_h), \phi\big),&\\ + \; \varepsilon \left((\partial_{tt}\, u_h, \phi) + \sum_{i=1}^{d} (A_i\; P\, \partial_{x_i}\, u_h, P\, \partial_{x_i}\, \phi) \right) &= 0 \quad \forall \phi \in S_k,\end{aligned} \tag{1.8}$$

where P is the L^2-projection operator onto S_k. Note that, when discretizing (1.7) in time with an explicit scheme, the computation of u_h will require the inversion of $d+1$ systems with the same mass matrix. The same procedure in (1.5) will require only the inversion of one mass matrix.

Based on the semidiscrete schemes one can devise various one-step implicit-explicit Runge–Kutta time discretizations [40, 4, 2, 3]. In the following sections we analyze the stability properties of semidiscrete as well as fully discrete schemes.

1.2. Stabilization by mesh refinement. Schemes (1.5) and (1.7) are indeed simple, but the relaxation mechanism alone does not provide the necessary stabilization required in the shock regions. Indeed, this is confirmed by coarse mesh numerical experiments; see section 6 and [4]. This also becomes evident by further examination of properties of the schemes. Consider the one-space dimensional ($d = 1$) system

$$\begin{aligned} &\partial_t u + \partial_x F(u) = 0, \quad x \in \mathbb{R},\ t > 0,\ \ u = u(x,t) \in \mathbb{R}^n, \\ &u(\cdot, 0) = u_0(\cdot) \end{aligned} \tag{1.9}$$

with u_0 of compact support and the associated finite element relaxation scheme. Following the argument in [4], it is seen that the effective equation of both schemes (1.5) and (1.7) in the case $n = 1,\ d = 1,\ q = 1$ is

$$\partial_t u + F(u)_x + \varepsilon\big[\partial_{tt} u - A\partial_{xx} u\big] + \beta\, h_{\rm loc}^2 F(u)_{xxx} = 0 \tag{1.10}$$

for some positive constant β. As expected, the finite element discretization induces a dispersion term which is linear in the flux variable. Applying the Chapman–Enskog expansion to (1.10) we obtain

$$\partial_t u + F(u)_x - \varepsilon \partial_x\big((c^2 - F'(u)^2)\,\partial_x u\big) + \beta\, h_{\rm loc}^2 F(u)_{xxx} = 0.$$

It is evident that to exclude approximations with oscillatory character near shocks or to avoid computing nonentropic solutions, the diffusion term should be dominant; see the relevant numerical example in section 6 and the literature on diffusion-dispersion approximations of conservation laws [28, 29]. This will enforce a condition of the form

$$h_{\rm loc} < o(\varepsilon), \tag{1.11}$$

where $h_{\rm loc}$ is the local mesh size close to the shock. On the other hand, the theoretical analysis in sections 2–4 provides convergence results under the slightly weaker condition

$$h_{\rm loc} < \gamma\, \varepsilon \tag{1.12}$$

for some constant γ. That is, the convergence results include even certain cases pertaining to nonclassical shocks. However, in practice typically mesh adaptivity selects the entropic solution, since it applies mesh refinement in a neighborhood of the shock. The extensive numerical experiments in [4, 2] and section 6 show that appropriate mesh refinement indeed stabilizes in a robust way the finite element relaxation schemes. Since the focus of this paper is the theoretical justification of the above schemes, we will not insist on the important problem of identifying appropriate mesh refinement strategies and refer to [4, 2, 3].

1.3. Stability and related properties. In what follows, we investigate the theoretical properties of the relaxation finite element schemes (1.5). It is shown that for a wide class of one-dimensional but also of multidimensional systems (1.1), the schemes are stable in the sense that they satisfy certain strong dissipation estimates; see Propositions 2.1, 2.3, 2.6, 3.1, 3.3, and 3.5. Similar estimates are satisfied by the relaxation model (1.2) [43, 18]. The strong dissipation estimates for relaxation approximations introduced in [43] are a basic tool in our analysis. In addition, nonstandard stability estimates for appropriate finite element projections are used in an essential way. The stability results are of interest since they justify the dissipative character of our schemes.

The stability estimates will also be used in conjunction with the compensated compactness framework to derive compactness conditions. Recall that a pair of functions $\eta = \eta(u)$, $q = q(u)$ are called an entropy-entropy flux pair (or entropy pair for short) if (η, q) solve the linear hyperbolic system

$$\nabla q = \nabla \eta \cdot \nabla F.$$

The existence and properties of entropy pairs have been extensively investigated (e.g., [15, 38]), and entropy pairs are used to describe the compactness properties of approximate solutions for certain one-dimensional systems of two conservation laws [42, 15, 38, 37].

In fact, we show that for the finite element relaxation scheme (1.5) with $d = 1$, the approximations u_h satisfy

$$\partial_t \, \eta(u_h) \, + \, \partial_x \, q(u_h) \ \subset \ \text{compact set of } H^{-1}_{\text{loc}}(\mathcal{O}).$$

This condition suffices to apply the compensated compactness program for certain one-dimensional equations and systems (see section 4) and to obtain convergence for semidiscrete or fully discrete finite element schemes. Similar results appear to hold for the fully conforming methods (1.7), (1.8), but their verification requires additional technical estimations. This is largely because the presence of the projection P in the one field equation (1.8) will result in extra error terms in the stability analysis. This case will not be pursued here.

The estimates derived in the following sections are rather complicated. To focus on the ideas and to present the material in a readable way, we have chosen to work step by step to distinguish the cases:

- semidiscrete schemes with symmetric flux F',
- semidiscrete schemes and the system admits a convex entropy function,
- fully discrete schemes with symmetric flux F',
- fully discrete schemes and the system admits a convex entropy function, and
- semidiscrete and fully discrete schemes for multidimensional systems that admit a convex entropy function.

In summary, the results provide theoretical support to the use of finite element relaxation schemes by establishing stability for a wide class of systems and convergence in various cases.

1.4. Error estimates for smooth solutions. Since the schemes are based on the discretization of model (1.2), in section 5 we address the problem of error estimates for relaxation approximations. We consider a system endowed with a convex entropy. Let u be a smooth solution of (1.1) defined on a maximal interval of existence, and let U_ε be the smooth solution of the relaxation approximation (1.2). We show that

$$\|U_\varepsilon(t) - u(t)\|_{L^2} \leq C(t,u)\,\varepsilon, \tag{1.13}$$

where the constant $C(t,u)$ depends on a strong norm of u and blows up at the critical time. The proof is based on a novel application of an idea of Dafermos [14, Thm. 5.2.1] to an error estimation. The difficulty posed by the relaxation approximation is handled by introducing a modified functional, corresponding to the relative entropy

$$H_R(u, U_\varepsilon) = \eta\big(U_\varepsilon + \varepsilon\partial_t(U_\varepsilon - u)\big) - \eta(u) - \eta'(u)\big(U_\varepsilon - u + \varepsilon\partial_t(U_\varepsilon - u)\big) \tag{1.14}$$

in the place of

$$H(u,w) = \eta(w) - \eta(u) - \eta'(u)(w-u) \tag{1.15}$$

used in [14]; see section 5 for details.

The finite element relaxation schemes are related to the central difference schemes of [33, 27]. One of their main common properties is that both schemes are Riemann solvers free and thus they combine high accuracy with simplicity. Finite element methods for hyperbolic conservation laws were considered in [21, 39, 22, 23, 19, 12, 10]. The theoretical properties of the streamline diffusion method were analyzed extensively (convergence, error estimates) in the scalar case [21, 39, 9]. The case of systems admitting entropy pairs is considered in [23] and it is shown that, for a streamline diffusion shock capturing method defined using the entropy variables, the bounded a.e. converging limits of approximations are weak entropy solutions of the system.

Finite element methods with discontinuous elements were proposed in [19] and [12]. In [12] stabilization is enforced by applying projection operators based on limiters. The above methods use piecewise polynomials of arbitrary degree and are formally of high order. Adaptive finite element methods based on a posteriori estimates have been considered in [22] for the ϵ-viscous approximation of one-dimensional systems of conservation laws. There exists a large literature on finite difference relaxation schemes; see, e.g., [20, 1, 25, 18] and [24] for relaxation schemes on unstructured grids.

The article is organized as follows. In section 2 we consider semidiscrete schemes and show stability and compactness of the dissipation measure for (i) case $d = 1$, F' is symmetric; (ii) case $d = 1$ and the system admits a convex entropy; and (iii) the multidimensional case. Section 3 is devoted to the analysis of implicit-explicit fully discrete schemes. The proofs are presented in a compact way, avoiding repetition of arguments already used in the semidiscrete case. In section 4 we discuss issues related to the application of compensated compactness to certain specific systems in order to conclude convergence of the schemes to a weak solution of (1.1). Section 5 is devoted to the error estimation between a smooth solution of (1.1) and the relaxation model (1.2). We conclude in section 6 with a discussion of implementation issues and present indicative examples reflecting the numerical performance of the method in two test cases.

2. Semidiscrete schemes: Stability estimates.

We start by showing that the scheme (1.5) admits a field equation that is in fact a standard finite element discretization of the conservation law perturbed by a wave operator.

LEMMA 2.1. *If u_h solves* (1.5)*, then it satisfies* (1.6).

Proof. Select $\psi = \partial_{x_i}\phi$, $\phi \in S_k$ in (1.5). Since $\psi \in V_{k-1}$ we have on summing with respect to i, $i = 1, \dots, d$,

$$\sum_{i=0}^{d}(\partial_t\, v_{h,i}, \partial_{x_i}\, \phi) \;+\; \sum_{i=1}^{d}(A_i\, \partial_{x_i}\, u_h, \partial_{x_i}\, \phi) = \;-\frac{1}{\varepsilon}\sum_{i=0}^{d}(v_{h,i} - F_i(u_h), \partial_{x_i}\, \phi).$$

Differentiating the first equation of (1.5) with respect to t we get

$$(\partial_{tt}\, u_h, \phi) - \sum_{j=1}^{d}(\partial_t v_{h,j},\; \partial_{x_j}\phi) \;=\; 0.$$

Hence,

$$\varepsilon(\partial_{tt}\, u_h, \phi) \;+\; \varepsilon\; \sum_{i=1}^{d}(A_i\, \partial_{x_i}\, u_h, \partial_{x_i}\, \phi) + \sum_{i=1}^{d}(v_{h,i}, \partial_{x_i}\, \phi) \;-\; \sum_{i=1}^{d}\big(F_i(u_h), \partial_{x_i}\, \phi\big) \;=\; 0.$$

Then by (1.5) we get the desired relation. □

In what follows, we establish stability properties for the finite element scheme (1.6). The stability estimates are proved consecutively for (i) case $d = 1$, F', symmetric; (ii) case $d = 1$, and the system admits a convex entropy; and (iii) the multidimensional case.

The one-dimensional semidiscrete finite element scheme takes the form

$$(2.1)\quad (\partial_t\, u_h, \phi) - \big(F(u_h), \partial_x\phi\big) + \varepsilon\, \Big((\partial_{tt}\, u_h, \phi) + (A\partial_x u_h, \partial_x\, \phi)\Big) = 0 \quad \forall \phi \in S_k.$$

For (2.1), we also prove compactness of the dissipation measure so as to apply the compensated compactness program and deduce convergence of the scheme in section 4. In the proof we use Murat's lemma [32].

LEMMA 2.2 (see Murat [32]). *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$ and $\{\phi_j\}$ a bounded sequence of $W^{-1,p}(\mathcal{O})$ for some $p > 2$. In addition let $\phi_j = \chi_j + \psi_j$, where $\{\chi_j\}$ belongs in a compact set of $H^{-1}(\mathcal{O})$ and $\{\psi_j\}$ belongs in a bounded set of the space of measures $M(\mathcal{O})$. Then $\{\phi_j\}$ belongs in a compact set of $H^{-1}(\mathcal{O})$.*

2.1. The case $d = 1$ and F' is symmetric. Let $\phi = u_h$ in (2.1) and use $\big(F(u_h), \partial_x u_h\big) = 0$ to get

$$(2.2)\quad \partial_t \left[\int_\Omega \left(\frac{1}{2}|u_h|^2 \;+\; \varepsilon\, u_h\, \partial_t u_h\right) dx\right] \;+\; \varepsilon \int_\Omega [A\partial_x\, u_h \cdot \partial_x\, u_h - (\partial_t u_h)^2] dx = 0.$$

To estimate $\varepsilon \int_\Omega (\partial_t u_h)^2 dx$ let $\phi = \partial_t u_h$ in (2.1). Then,

$$(2.3)\quad \begin{aligned} &\|\partial_t\, u_h\|^2_{L^2} + \;(F'(u_h)\, \partial_x\, u_h, \partial_t\, u_h) \\ &\qquad +\; \varepsilon \frac{1}{2}\partial_t \|\partial_t\, u_h\|^2_{L^2} +\; \varepsilon \frac{1}{2}\partial_t (A\partial_x\, u_h, \partial_x\, u_h) = 0. \end{aligned}$$

Adding (2.2) with 2ε times (2.3) yields

$$\begin{aligned} &\frac{1}{2}\,\partial_t\, \|u_h + \varepsilon\partial_t\, u_h\|^2_{L^2} + \varepsilon\, (A\partial_x\, u_h, \partial_x\, u_h) \;+\; 2\varepsilon(F'(u_h)\, \partial_x\, u_h, \partial_t\, u_h) \\ &\qquad + \varepsilon\, \|\partial_t\, u_h\|^2_{L^2} \;+\; \frac{1}{2}\, \varepsilon^2\, \partial_t \Big\{ \|\partial_t\, u_h\|^2_{L^2} + 2(A\partial_x\, u_h, \partial_x\, u_h)\Big\} \;=\; 0. \end{aligned}$$

Since F' is symmetric, we have

$$\|\partial_x F(u_h)\|^2_{L^2} = \big({F'}^2(u_h)\partial_x u_h, \partial_x u_h\big),$$

and we obtain

$$\begin{aligned} &\frac{1}{2}\,\partial_t \Big\{ \|u_h + \varepsilon\partial_t\, u_h\|^2_{L^2} + \varepsilon^2 \|\partial_t u_h\|^2_{L^2} + 2\varepsilon^2 (A\partial_x\, u_h, \partial_x\, u_h)\Big\} \\ &\qquad + \varepsilon \|\partial_t\, u_h + \partial_x F(u_h)\|^2_{L^2} + \varepsilon\big([A - {F'}^2(u_h)]\partial_x u_h, \partial_x u_h\big) = 0. \end{aligned}$$

We conclude with the following.

PROPOSITION 2.1. *Assume that $F'(u), A$ are symmetric and satisfy for some $\nu > 0$*

$$A - F'(u)^2 \geq \nu\, I, \qquad u \in \mathbb{R}^n. \tag{2.4}$$

Then the finite element approximation (2.1) *satisfies*

$$\begin{aligned}
&\int_\Omega \Big(|u_h + \varepsilon\, \partial_t\, u_h|^2 \;+\; \varepsilon^2 |\partial_t\, u_h|^2 \;+\; 2\varepsilon^2 A \partial_x\, u_h \;\cdot\; \partial_x\, u_h \Big) \\
&\qquad + 2 \int_0^t \int_\Omega \Big(\varepsilon |\partial_t u_h + F'(u_h) \partial_x\, u_h|^2 \;+\; \varepsilon\, \nu |\partial_x\, u_h|^2 \Big) \\
&\qquad \leq \int_\Omega |u_h^0 + \varepsilon\, \partial_t\, u_h(0)|^2 + \varepsilon^2 |\partial_t\, u_h(0)|^2 + 2\varepsilon^2 A \partial_x\, u_h^0 \cdot \partial_x\, u_h^0 \;=: C(u_h^0).
\end{aligned}$$

In what follows we prove the next proposition.

PROPOSITION 2.2. *Let (η, q) be an entropy pair satisfying*

$$\|\eta\|_{L^\infty},\; \|q\|_{L^\infty},\; \|\eta'\|_{L^\infty},\; \|\eta''\|_{L^\infty} \;\leq\; C.$$

Then, for $h \;\leq\; \gamma\, \varepsilon$, there holds

$$\eta(u_h)_t \;+\; q(u_h)_x \text{ lies in a compact set of } H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+).$$

Proof. Let (η, q) be an entropy pair and $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$ a test function, and $\operatorname{supp} \phi \subset \tilde{\Omega} \times [0, \tilde{T}] =: Q$. We denote by $\Pi \;:\; L^2(\Omega) \;\to\; S_k$ a projection operator onto the finite element space of u_h to be determined later. Using the definition of the scheme we have

$$\begin{aligned}
\Big(\eta(u_h)_t \;+\; q(u_h)_x, \phi \Big) &= \Big(\eta'(u_h) \big[u_{h,t} \;+\; F'(u_h) u_{h,x} \big], \phi \Big) \\
&= \Big(\big[u_{h,t} + F'(u_h) u_{h,x} \big], \Pi\big(\eta'(u_h)\, \phi \big) \Big) \\
&\quad + \Big(\big[u_{h,t} + F'(u_h) u_{h,x} \big], \eta'(u_h) \phi \;-\; \Pi(\eta'(u_h) \phi) \Big) \\
&= -\, \varepsilon \Big(A \partial_x\, u_h, \big[\Pi\big(\eta'(u_h) \phi \big) \big]_x \Big) - \varepsilon \Big(u_{h,tt}, \Pi\big(\eta'(u_h) \phi \big) \Big) \\
&\quad + \Big(\big[u_{h,t} + F'(u_h) u_{h,x} \big],\; \eta'(u_h) \phi - \Pi\big(\eta'(u_h) \phi \big) \Big).
\end{aligned} \tag{2.5}$$

We select now $\Pi \;:\; L^2(\Omega) \;\to\; S_k$ to be the L^2-projection onto S_k. Π satisfies

$$(\Pi \omega, \phi) \;=\; (\omega, \phi) \qquad \forall \phi \in S_k,\; \omega \in L^2(\Omega), \tag{2.6}$$

$$\|\Pi \omega - \omega\|_{L^2(\Omega)} \;=\; \inf_{\chi \in S_k} \|\omega - \chi\|_{L^2(\Omega)} \;\leq\; C\, h\, \|\omega_x\|_{L^2(\Omega)}, \quad \omega \in H^1(\Omega), \tag{2.7}$$

as well as the stability estimate [13]

$$\|(\Pi \omega)_x\|_{L^2(\Omega)} \;\leq\; C \|\omega_x\|_{L^2(\Omega)}, \quad \omega \in H^1(\Omega). \tag{2.8}$$

We are ready to bound the terms in the right-hand side of (2.5). Indeed, (2.8) implies

$$\begin{aligned}
&\varepsilon \Big| \Big(A\, u_{h,x}, \big[\Pi\big(\eta'(u_h) \phi \big) \big]_x \Big) \Big| \leq \varepsilon\, C \|u_{h,x}\|_{L^2(\Omega)}\, \big\| \big(\eta'(u_h) \phi \big)_x \big\|_{L^2(\Omega)} \\
&\quad \leq C \Big(\varepsilon \int_\Omega |u_{h,x}|^2 \Big) \|\eta''\|_{L^\infty(\Omega)} \|\phi\|_{C^0(\Omega)} + \varepsilon^{1/2} C \Big(\varepsilon \int_\Omega |u_{h,x}|^2 \Big)^{1/2} \|\eta'\|_{L^\infty(\Omega)} \|\phi_x\|_{L^2(\Omega)}.
\end{aligned} \tag{2.9}$$

Next, since $u_{h,tt} \in S_k$ and by (2.6),

$$- \varepsilon \int_0^t \int_\Omega u_{h,tt}\, \Pi\Big(\eta'(u_h)\phi\Big) dxds = -\varepsilon \int_0^t \int_\Omega u_{h,tt}\, \eta'(u_h)\phi dxds$$
$$= \varepsilon \int_0^t \int_\Omega u_{h,t} \big(\eta'(u_h)\phi\big)_t dxds + \varepsilon \int_\Omega u_{h,t}\, \eta'(u_h)\phi\Big|_{s=0} dx - \varepsilon \int_\Omega u_{h,t}\, \eta'(u_h)\phi\Big|_{s=t} dx.$$

By Proposition 2.1 we have

$$\varepsilon \Big| \int_\Omega u_{h,t}\, \eta'(u_h)\phi(t) dx \Big| \le \varepsilon \Big(\int_\Omega u_{h,t}^2 \Big)^{1/2} \|\eta'\|_{L^\infty(\Omega)}\, \|\phi\|_{C^0(\Omega)}\, m(\Omega)^{1/2} \le C_\Omega \|\phi\|_{C^0(\Omega)}, \tag{2.10}$$

and as before

$$\varepsilon \Big| \int_0^t \int_\Omega u_{h,t} \big(\eta'(u_h)\phi\big)_t dxdt \Big| \le C\Big(\varepsilon \int_0^t \int_\Omega |u_{h,t}|^2 \Big) \|\eta''\|_{L^\infty(Q)}\, \|\phi\|_{C^0(Q)} + \varepsilon^{1/2} \Big(\varepsilon \int_0^t \int_\Omega |u_{h,t}|^2 \Big)^{1/2} \|\eta'\|_{L^\infty(Q)}\, \|\phi_t\|_{L^2(Q)}. \tag{2.11}$$

To estimate the last term in (2.5), note that $\eta'(u_h)\phi \in H^1(\Omega)$ and thus

$$\|\eta'(u_h)\phi - \Pi\big(\eta'(u_h)\phi\big)\|_{L^2(\Omega)} \le Ch \|\big(\eta'(u_h)\phi\big)_x\|_{L^2(\Omega)} + Ch\|\eta'(u_h)\phi_x\|_{L^2(\Omega)}$$
$$\le Ch\|\eta''\|_{L^\infty(\Omega)}\, \|u_{h,x}\|_{L^2(\Omega)}\, \|\phi\|_{C^0(\Omega)} + Ch\|\eta'\|_{L^\infty(\Omega)}\, \|\phi_x\|_{L^2(\Omega)}.$$

By (2.4) we have $\|F'(u_h)^2\|_{L^\infty(\Omega)} \le C$; therefore

$$\begin{aligned} &\Big| \Big(\big[u_{h,t} + F'(u_h)u_{h,x}\big],\ \eta'(u_h)\phi - \Pi\big(\eta'(u_h)\phi\big) \Big) \Big| \\ &\quad \le C\Big(h \int_\Omega (|u_{h,t}|^2 dx + |\partial_x u_h|^2) dx \Big) \|\phi\|_{C^0(\Omega)} \\ &\quad + h \left(\int_\Omega (|u_{h,t}|^2 + |\partial_x u_h|^2)\, dx \right)^{1/2} \|\phi_x\|_{L^2(\Omega)}. \end{aligned} \tag{2.12}$$

Combining (2.9)–(2.12) and using Murat's Lemma 2.2 (in our case, $\chi_h \to 0$ in H^{-1} and is thus precompact in H^{-1}), we complete the proof. ∎

2.2. The case $d = 1$, and the system admits a convex entropy. The case that F' is not necessarily symmetric but the system is equipped with a convex entropy η is examined next. In this case the system is symmetrizable. The finite element approximations (1.5) enjoy the same a priori bounds with the continuous solution of the relaxation model considered in [43]. Indeed, the following proposition holds.

PROPOSITION 2.3. *Let (1.9) be equipped with a strictly convex entropy $\eta(u)$ satisfying for some $\alpha > 0$*

$$\frac{1}{\alpha} I \le \eta''(u) \le \alpha\, I, \quad u \in \mathbb{R}^n. \tag{2.13}$$

Assume for some $M > 0$ we have $|F'(u)| \le M$ for $u \in \mathbb{R}^n$ and that the positive definite, symmetric matrix A is selected to satisfy, for $\overline{\alpha} = 2\alpha \max\{\beta, 1\}$, β as in (2.22) and some $\nu > 0$,

$$\frac{1}{2}\big((\eta''(u)A)^T + \eta''(u)A\big) - \overline{\alpha} F'(u)^T F'(u) \ge \nu\, I \qquad \text{for } u \in \mathbb{R}^n. \tag{2.14}$$

Then there is $\gamma = \gamma(\alpha, \beta, M, \nu) > 0$ *such that, for*

$$h \leq \gamma \varepsilon \tag{2.15}$$

and for some positive constants c_1, c_2, *and* c_3, *the finite element approximation* (2.1) *satisfies the stability estimate*

$$\begin{aligned} &\int_\Omega \Big(\eta(u_h + \varepsilon\, \partial_t\, u_h) + \varepsilon^2 c_1 \big[|\partial_t\, u_h|^2 + A\partial_x\, u_h \cdot \partial_x\, u_h \big] \Big) dx \\ &\quad + \varepsilon c_2 \int_0^t \int_\Omega \Big(|\partial_t u_h + F'(u_h)\partial_x\, u_h|^2 + |\partial_x\, u_h|^2 + |\partial_t u_h|^2 \Big) dx dt \\ &\quad \leq \int_\Omega \Big(\eta(u_h^0 + \varepsilon\, \partial_t\, u_h(0)) + \varepsilon^2 c_3 \big[|\partial_t\, u_h(0)|^2 + A\partial_x\, u_h^0 \cdot \partial_x\, u_h^0 \big] \Big) dx. \end{aligned} \tag{2.16}$$

Remark 2.1. We are interested here in data and associated finite element approximations u_h that are of compact support. It is thus natural to normalize η so that $\eta(0) = 0$ and $\eta'(0) = 0$. This can always be achieved, because if (η, q) is an entropy pair, then

$$\eta(u) - \eta(0) - \eta'(0)u, \quad q(u) - q(0) - \eta'(0)(F(u) - F(0))$$

is also an entropy pair. In view of (2.13), the normalized η is equivalent to the Euclidean norm, $\eta(u) \sim |u|^2$. Thus the stability framework in Proposition 2.3 is that of L^2.

Using the stability estimate, it is easy to see that strong convergence of the finite element approximations gives a weak solution that satisfies the integral version of the entropy inequality.

PROPOSITION 2.4. *Under the hypotheses of Proposition* 2.3, *if*

$$u_h \to u \quad \text{in } L^2_{x,t} \text{ and a.e.,} \tag{2.17}$$

then u *is a weak solution of* (1.9) *that satisfies*

$$\int_\Omega \eta(u(x,t))dx \leq \int_\Omega \eta(u^0(x))dx \qquad \text{for a.e. } t. \tag{2.18}$$

Proof. We assume with no loss of generality that $F(0) = 0$ and note that $|F(u)| \leq M|u|$. Let $u^0 \in H_0^1$ and be of compact support, let $v^0 = F(u^0) \in H_0^1$ and be of compact support, and define the approximations $u_h^0 \in S_k$ and $v_h^0 \in V_{k-1}$ defined by $v_h^0 = \Pi_{V_{k-1}} F(u_h^0)$ with $\Pi_{V_{k-1}}$ the L^2-projection. Let $u_h = u_h(x,t)$, $v_h = v_h(x,t)$ be the solution of the semidiscrete scheme. Note that $\partial_t u_h(0) = \Pi_{S_k} \partial_x F(u_h^0)$, where Π_{S_k} is the L^2-projection onto S_k.

For $\phi(x) \in S_k$ and $\theta(t) \in C_c^\infty([0,\infty))$ we have

$$\begin{aligned} &-\int_0^t \int_\Omega [u_h \phi \partial_t \theta + F(u_h)\partial_x \phi \theta - \epsilon A \partial_x u_h \cdot \partial_x \phi \theta + \epsilon \partial_t u_h \cdot \phi \partial_t \theta] dx dt \\ &\quad - \int_\Omega (u_h^0 \phi \theta(0) + \epsilon \partial_t u_h(0) \phi \theta(0)) dx = 0. \end{aligned} \tag{2.19}$$

Note that

$$\begin{aligned} &u_h \to u, \;\; F(u_h) \to F(u) \quad \text{in } L^2_{x,t} \text{ and a.e.,} \\ &u_h^0 \to u^0, \;\; \varepsilon \partial_t u_h(0) \to 0 \quad \text{in } L^2_x \text{ and (along a subsequence) a.e.,} \\ &\varepsilon^{\frac{1}{2}} \|\partial_x u_h\|_{L^2_{x,t}} + \varepsilon^{\frac{1}{2}} \|\partial_t u_h\|_{L^2_{x,t}} \leq O(1). \end{aligned}$$

Using that tensor products $\phi(x) \otimes \theta(t)$, $\phi \in S_k$, $\theta \in C_c^\infty([0,\infty))$ are dense as $h \to 0$ in $C^2(\bar{\Omega})$ for Ω bounded, we pass to the limit in (2.19) and obtain that u is a weak solution of (1.9). Using Fatou's lemma, we pass to the limit ε, $h \to 0$ in (2.16) to deduce

$$\int_\Omega \eta(u(x,t))\,dx \le \liminf_{h\to 0,\,\varepsilon\to 0} \int_\Omega \eta(u_h + \varepsilon\partial_t u_h)\,dx \le \int_\Omega \eta(u_0(x))\,dx$$

and conclude. □

To show the stability estimate we use the elliptic projection operator onto S_k and its approximation and stability properties. To this end let $P_1 : H_0^1 \to S_k$ be the Ritz (elliptic) projection defined by

$$(A\partial_x\, P_1 v, \partial_x\phi) = (A\partial_x\, v, \partial_x\phi) \qquad \forall\phi \in S_k,\ v \in H_0^1. \tag{2.20}$$

It is a standard result that P_1 satisfies

$$\begin{aligned} \|P_1\omega - \omega\|_{L^2(\Omega)} &\le C\,h\,\|\omega_x\|_{L^2(\Omega)}, \quad \omega \in H_0^1, \\ \|(P_1\omega)_x\|_{L^2(\Omega)} &\le C\|\omega_x\|_{L^2(\Omega)}, \quad \omega \in H_0^1. \end{aligned} \tag{2.21}$$

The second bound is a direct consequence of the definition and the first is obtained by a standard duality argument using once more the second bound (see [7, Thm. 5.4.8]). The following nonstandard stability property of P_1 will be crucial in the proof of Proposition 2.3. It uses in an essential way the stability analysis of the finite element method by mesh-dependent norms due to Babuška and Osborn [5].

LEMMA 2.3. *Let η be a strictly convex entropy and $v_h \in S_k$. Under hypothesis* (2.13)*, there exists a positive constant β such that*

$$(v_h, P_1\,[\,\eta''(w)(v_h)\,]\,) \;\le\; \beta\,\|\eta''(w)\|_{L^\infty(\Omega)}\,\|v_h\|^2_{L^2(\Omega)} \quad \forall w \in S_k. \tag{2.22}$$

Proof. It is known that P_1 is not stable with respect to $L^2(\Omega)$ [5]. Its stability with respect to the mesh-dependent L^2-like norm

$$\|v\|_{0,h,\Omega} = \left(\|v\|^2_{L^2(\Omega)} + \sum_j \delta_j |v(x_j)|^2\right)^{1/2}, \tag{2.23}$$

where x_j are the nodes of the partition and $\delta_j = (x_{j+1} - x_{j-1})/2$ is as shown in [5], and

$$\|P_1\,v\|_{0,h,\Omega} \;\le\; \beta_1\|v\|_{0,h,\Omega}, \tag{2.24}$$

where β_1 is a positive constant independent of h. Thus, (2.24) implies

$$\|P_1\,[\,\eta''(w)(v_h)\,]\,\|_{L^2(\Omega)} \;\le\; \beta_1\|\eta''(w)\|_{L^\infty(\Omega)}\,\|v_h\|_{0,h,\Omega}. \tag{2.25}$$

But in the finite element space local inverse inequalities imply

$$\|v_h\|_{0,h,\Omega} \;\le\; \beta_2\|v_h\|_{L^2(\Omega)} \qquad \forall v_h \in S_k, \tag{2.26}$$

with β_2 independent of h [7, 5]. Therefore, (2.22) follows with $\beta = \beta_1\,\beta_2$. □

Proof of Proposition 2.3. The finite element approximation u_h satisfies (2.1). Setting $\phi = P_1\,\eta'(u_h)$ and using (2.20), we obtain after a rearrangement

$$\begin{aligned}(\partial_t\, u_h, \eta'(u_h)) + \big(\partial_x F(u_h), \eta'(u_h)\big) &+ \varepsilon(\partial_{tt} u_h, P_1\eta'(u_h)) + \varepsilon(A\partial_x u_h, \partial_x \eta'(u_h)) \\ &= (\partial_t\, u_h, \eta'(u_h) - P_1\,\eta'(u_h)) + \big(\partial_x F(u_h), \eta'(u_h) - P_1\,\eta'(u_h)\big) \\ &=: Z_1 + Z_2.\end{aligned} \tag{2.27}$$

The terms in the right-hand side will be estimated in what follows. First we examine the stability properties of the left-hand side. Since P_1 commutes with time differentiation,

$$\begin{aligned}\varepsilon(\partial_{tt}\, u_h, P_1\,\eta'(u_h)) &= \varepsilon\partial_t\,(\partial_t\, u_h, P_1\,\eta'(u_h)) - \varepsilon(\partial_t\, u_h, P_1\,[\,\eta''(u_h)\partial_t\, u_h\,]) \\ &= \varepsilon\partial_t\,(\partial_t\, u_h, \eta'(u_h)) - \varepsilon(\partial_t\, u_h, P_1\,[\,\eta''(u_h)\partial_t\, u_h\,]) \\ &\quad - \varepsilon\partial_t\,(\partial_t\, u_h, \eta'(u_h) - P_1\eta'(u_h)).\end{aligned} \tag{2.28}$$

We thus have

$$\begin{aligned}\partial_t \int_\Omega \eta(u_h) + \int_\Omega \partial_x q(u_h) &+ \varepsilon\partial_t\,(\partial_t\, u_h, \eta'(u_h)) \\ &+ \varepsilon(A\partial_x\, u_h, \eta''(u_h)\partial_x\, u_h) - \varepsilon(\partial_t\, u_h, P_1\,[\,\eta''(u_h)\partial_t\, u_h\,]) \\ &= Z_1 + Z_2 + Z_3,\end{aligned} \tag{2.29}$$

where the new term Z_3 is given by

$$Z_3 = \varepsilon\partial_t\,(\partial_t\, u_h, \eta'(u_h) - P_1\eta'(u_h)) = \varepsilon\partial_t Z_1. \tag{2.30}$$

As in [43] the following identity will be important:

$$\begin{aligned}\int_\Omega \eta(u_h + \varepsilon\partial_t\, u_h)dx &= \int_\Omega \eta(u_h)dx + \varepsilon(\eta'(u_h), \partial_t\, u_h) \\ &\quad + \varepsilon^2 \left(\partial_t\, u_h, \left\{\int_0^1 \int_0^s \eta''(u_h + \varepsilon\,\tau\partial_t\, u_h)d\tau ds\right\}\partial_t\, u_h\right).\end{aligned} \tag{2.31}$$

It is evident that we need to estimate $\varepsilon(\partial_t u_h, P_1[\,\eta''(u_h)\partial_t u_h\,])$. This is done by Lemma 2.3, which gives

$$\varepsilon|(\partial_t\, u_h, P_1\,[\,\eta''(u_h)\partial_t\, u_h\,])| \le \varepsilon\beta\,\|\eta''(u_h)\|_{L^\infty(\Omega)}\,\|\partial_t\, u_h\|^2_{L^2(\Omega)}. \tag{2.32}$$

We proceed to handle $\varepsilon\int_\Omega(\partial_t u_h)^2 dx$. Observe that setting $\phi = \partial_t u_h$ in (2.1) gives

(2.33)

$$\|\partial_t\, u_h\|^2_{L^2(\Omega)} + (F'(u_h)\partial_x\, u_h, \partial_t\, u_h) + \varepsilon\frac{1}{2}\partial_t\|\partial_t\, u_h\|^2_{L^2(\Omega)} + \varepsilon\frac{1}{2}\partial_t(A\partial_x\, u_h, \partial_x\, u_h) = 0.$$

Next, define

$$\begin{aligned}\overline{\beta} &= \beta\,\|\eta''(u_h)\|_{L^\infty(\Omega)}, \\ \overline{\eta''} &= \left\{\int_0^1 \int_0^s \eta''(u_h + \varepsilon\,\tau\partial_t\, u_h)d\tau ds\right\}, \\ \overline{\alpha} &= \max\{2\overline{\beta}, 2\alpha\}\end{aligned} \tag{2.34}$$

and note that $\overline{\beta} = \beta\alpha$, $\overline{\alpha} = 2\alpha \max\{1, \beta\}$. After summing (2.29) with $2\varepsilon\,\overline{\alpha}$ times (2.33), we arrive at

$$\begin{aligned}
&\partial_t \int_\Omega \Big(\eta(u_h + \varepsilon\partial_t u_h) + \varepsilon^2 \partial_t u_h \cdot \{\overline{\alpha}\, I - \overline{\eta''}\} \partial_t u_h + \varepsilon^2 \overline{\alpha} A \partial_x\, u_h \cdot \partial_x u_h\Big) dx - \varepsilon \partial_t Z_1 \\
&\qquad + \varepsilon\,(\overline{\alpha} - \overline{\beta}) \|\partial_t\, u_h\|^2_{L^2(\Omega)} + \varepsilon\overline{\alpha} \|\partial_t\, u_h\|^2_{L^2(\Omega)} + 2\varepsilon\overline{\alpha}(F'(u_h)\partial_x u_h, \partial_t u_h\,) \\
&\qquad + \varepsilon(A\partial_x\, u_h, \eta''(u_h)\partial_x\, u_h) \le Z_1 + Z_2.
\end{aligned} \tag{2.35}$$

But since

$$\begin{aligned}
&\|\partial_t\, u_h\|^2_{L^2(\Omega)} + 2(F'(u_h)\; \partial_x\, u_h, \partial_t\, u_h\,) \\
&\quad = \|\partial_t u_h + F'(u_h)\partial_x u_h\|^2_{L^2(\Omega)} - (F'(u_h)^T F'(u_h)\partial_x u_h, \partial_x\, u_h)
\end{aligned} \tag{2.36}$$

and

$$(A\partial_x\, u_h, \eta''(u_h)\partial_x\, u_h) = \frac{1}{2}\,\Big(\big(\eta''(u_h)A + (\eta''(u_h)A)^T\big)\partial_x\, u_h, \partial_x\, u_h\Big),$$

we conclude by (2.13) and (2.14) that

$$\begin{aligned}
&\partial_t \Big\{ \int_\Omega \eta(u_h + \varepsilon\partial_t u_h) dx + \varepsilon^2\overline{\alpha}\|\partial_t u_h\|^2_{L^2(\Omega)} + \varepsilon^2\overline{\alpha}(A\partial_x u_h, \partial_x u_h) - \varepsilon Z_1 \Big\} \\
&\qquad + \varepsilon\overline{\beta}\|\partial_t\, u_h\|^2_{L^2(\Omega)} + \varepsilon\,\overline{\alpha}\,\|\partial_t\, u_h + F'(u_h)\;\partial_x\, u_h\|^2_{L^2(\Omega)} + \varepsilon\nu\,\|\partial_x\, u_h\|^2_{L^2(\Omega)} \\
&\qquad \le Z_1 + Z_2.
\end{aligned} \tag{2.37}$$

At this point $\overline{\alpha}$, $\overline{\beta}$, and ν are fixed. We now turn to the estimation of the Z_i. Observe that, by (2.13) and (2.21),

$$\begin{aligned}
Z_1 &= (\partial_t\, u_h, \eta'(u_h) - P_1\, \eta'(u_h)) \\
&\le C\,h\,\|\partial_t\, u_h\|_{L^2(\Omega)}\;\|\partial_x\; \eta'(u_h)\|_{L^2(\Omega)} \\
&\le C\,h\,\|\partial_t\, u_h\|_{L^2(\Omega)}\;\|\eta''(u_h)\|_{L^\infty(\Omega)}\;\|\partial_x\, u_h\|_{L^2(\Omega)} \\
&\le Ch\,\alpha\,\|\partial_t\, u_h\|_{L^2(\Omega)}\;\|\partial_x\, u_h\|_{L^2(\Omega)},
\end{aligned} \tag{2.38}$$

while

$$\begin{aligned}
Z_2 &= \big(\partial_x F(u_h), \eta'(u_h) - P_1\, \eta'(u_h)\big) \\
&\le C\,h\,\|\partial_x\, u_h\|_{L^2(\Omega)}\;\|F'(u_h)\|_{L^\infty(\Omega)}\;\;\|\eta''(u_h)\|_{L^\infty(\Omega)}\;\|\partial_x\, u_h\|_{L^2(\Omega)} \\
&\le C\,h\,\alpha\,M\,\|\partial_x\, u_h\|^2_{L^2(\Omega)}.
\end{aligned} \tag{2.39}$$

Next, we select h so that (i) the quadratic form in the first term of (2.37) is positive definite, and (ii) the terms Z_1 and Z_2 on the right of (2.37) can be absorbed to the left. This can be done provided $h \le \gamma\varepsilon$ for some $\gamma = \gamma(\alpha, \beta, M, \nu)$ positive and small. This gives (2.16) and concludes the proof. ∎

The compactness of the dissipation measure for the scheme is obtained by an argument similar to that in the symmetric case.

Proposition 2.5. *For entropy pairs* (η, q) *satisfying*

$$\|\eta\|_{L^\infty},\; \|q\|_{L^\infty},\; \|\eta'\|_{L^\infty},\; \|\eta''\|_{L^\infty} \;\le\; C \tag{2.40}$$

and for $h \le \gamma\,\varepsilon$,

$$\eta(u_h)_t + q(u_h)_x \quad \textit{lies in a compact set of } H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{2.41}$$

Remark 2.2. Proposition 2.5 and the analogous statement for the symmetric case (Proposition 2.2) state that for entropy pairs satisfying (2.40) the entropy dissipation measure is controlled. They are used in section 4 to prove compactness of relaxation finite element approximations for the system (4.1). We note that entropy pairs (η, q) that satisfy (2.40) are constructed in [38] for the system (4.1) under hypotheses (4.3)–(4.4).

2.3. The multidimensional case. Next we consider multidimensional systems (1.1) for which the system is endowed with a uniformly convex entropy η. Let $(q_1, \dots, q_d)$ be the associated entropy flux, and

$$\begin{aligned} q_i'(u) &= \eta'(u)\,F_i'(u), \quad i=1,\dots,d,\\ \eta''(u)\,F_i'(u) &= F_i'(u)^T\,\eta''(u), \quad i=1,\dots,d. \end{aligned} \tag{2.42}$$

Still, in this case the finite element approximations (1.5) satisfy similar a priori bounds with the one-dimensional case, provided that each A_i is chosen to satisfy certain subcharacteristic conditions.

PROPOSITION 2.6. *Assume that* (1.1) *is equipped with a strictly convex entropy* $\eta(u)$ *that satisfies for some* $\alpha > 0$

$$\frac{1}{\alpha}\,I \le \eta''(v) \le \alpha\,I, \quad v\in\mathbb{R}^n; \tag{2.43}$$

let $\overline{\alpha} = 2\alpha\max\{1,\beta\}$ *with* β *as in* (2.46), *and assume that the symmetric, positive definite matrices* A_i *satisfy, for some* $\nu > 0$,

$$\sum_{j=1}^d \frac{1}{2}\big(A_j\eta''(v) + (A_j\eta''(v))^T\big)\xi_j\cdot\xi_j - \overline{\alpha}\left|\sum_{j=1}^d F_j'(v)\xi_j\right|^2 \ge \nu\sum_{j=1}^d|\xi_j|^2 \qquad \forall \xi_1,\dots,\xi_d\in\mathbb{R}^n,\ v\in\mathbb{R}^n. \tag{2.44}$$

If $h \le \gamma\varepsilon$ *for some* $\gamma > 0$, *then the finite element approximations* (1.5) *satisfy, for some* $c_1,\, c_2 > 0$, *the stability estimate*

$$\begin{aligned} &\int_\Omega\left(\eta(u_h+\varepsilon\partial_t u_h) + \varepsilon^2 c_1\left[|\partial_t u_h|^2 + \sum_{i=1}^d A_i\partial_{x_i}u_h\cdot\partial_{x_i}u_h\right]\right)\\ &\quad + \varepsilon c_2\int_0^t\int_\Omega\left(\left|\partial_t u_h + \sum_{i=1}^d F_i'(u_h)\partial_x u_h\right|^2 + \sum_{i=1}^d|\partial_{x_i}u_h|^2 + |\partial_t u_h|^2\right)\\ &\quad \le C(u_h^0, \partial_t u_h(0)\,). \end{aligned}$$

The proof is entirely similar to the one-dimensional case presented before and therefore it will be omitted. Still, an essential tool in the analysis will be the elliptic projection $P_1 : H^1 \to S_k$ defined by

$$\sum_{i=1}^d (A_i\partial_{x_i}\ P_1 v, \partial_{x_i}\phi) = \sum_{i=1}^d (A_i\partial_{x_i}\ v, \partial_{x_i}\phi) \qquad \forall\phi\in S_k. \tag{2.45}$$

The multidimensional analogue of Lemma 2.3 still holds:

$$(v_h, P_1\,[\,\eta''(u_h)(v_h)\,]\,) \;\le\; \beta\,\|\eta''(w)\|_{L^\infty(\Omega)}\,\|v_h\|^2_{L^2(\Omega)}. \tag{2.46}$$

Its proof is based on the stability analysis of the finite element method by mesh-dependent norms [6]; see [16] for related results on stability of the elliptic projection in $L^2(\Omega)$. The quasi-uniformity assumption on the mesh in [6] needed to verify (2.24) can be relaxed along the lines of arguments presented in [16].

3. Fully discrete schemes. There are many alternative ways to perform the time discretization of (1.5) at the discrete time nodes $0, \kappa, 2\kappa, \dots$. In this section we consider a simple implicit-explicit time discretization. Seek $(u_h^n, v_{h,1}^n, \dots, v_{h,d}^n) \in S_k \times V_{k-1}^d, n = 0, 1, \dots,$

$$\begin{aligned} &\left(\frac{u_h^{n+1} - u_h^n}{\kappa},\ \phi\right) - \sum_{i=1}^d (v_{h,i}^n,\ \partial_{x_i}\phi) = 0 \quad \forall \phi \in S_k,\\ &\left(\frac{v_{h,i}^{n+1} - v_{h,i}^n}{\kappa},\ \psi\right) + (A_i\partial_{x_i}u_h^{n+1},\ \psi) = -\frac{1}{\varepsilon}\big(v_{h,i}^{n+1} - F_i(u_h^{n+1}),\ \psi\big),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \psi \in V_{k-1}, \quad i = 1, \dots, d, \end{aligned} \tag{3.1}$$

where $u_h^0 = u_0,\ v_{h,i}^0 = F_i(u_0)$, and $i = 1, \dots, d$.

When $d = 1$, the scheme takes the form

$$\begin{aligned} &\left(\frac{u_h^{n+1} - u_h^n}{\kappa},\ \phi\right) - (v_h^n,\ \partial_x\phi) = 0 \quad \forall \phi \in S_k,\\ &\left(\frac{v_h^{n+1} - v_h^n}{\kappa},\ \psi\right) + (A\partial_x u_h^{n+1},\ \psi) = -\frac{1}{\varepsilon}\big(v_h^{n+1} - F(u_h^{n+1}),\ \psi\big) \quad \forall \psi \in V_{k-1}. \end{aligned} \tag{3.2}$$

3.1. Properties of the scheme. For any sequence $\{\,Y^n\} \subset L^2(\Omega)$, define the operators $\overline{\partial}_t,\ \overline{\partial}_{tt}$:

$$\overline{\partial}_t Y^n :=\ \frac{1}{\kappa}(Y^{n+1} - Y^n), \qquad \overline{\partial}_{tt} Y^n :=\ \overline{\partial}_t\overline{\partial}_t Y^n.$$

Then the centered difference quotient that corresponds to the second time derivative at t^n is

$$\overline{\partial}_{tt} Y^{n-1} = \frac{1}{\kappa^2}(Y^{n+1} - 2Y^n + Y^{n-1}).$$

The following properties will prove useful (L^2 stands for $L^2(\Omega)$):

$$\begin{aligned} (\overline{\partial}_t Y^n,\ Y^{n+1}) &= \frac{1}{2\kappa}\big[\,\|\,Y^{n+1}\|^2_{L^2} - \|\,Y^n\|^2_{L^2} + \|\,Y^{n+1} - Y^n\|^2_{L^2}\big]\\ &= \frac{1}{2}\big[\,\overline{\partial}_t\|\,Y^n\|^2_{L^2} + \kappa\|\,\overline{\partial}_t Y^n\|^2_{L^2}\big], \end{aligned} \tag{3.3}$$

$$(\overline{\partial}_t Y^n,\ Y^n) = \frac{1}{2}\big[\,\overline{\partial}_t\|\,Y^n\|^2_{L^2} - \kappa\|\,\overline{\partial}_t Y^n\|^2_{L^2}\big], \tag{3.4}$$

$$\begin{aligned} (\overline{\partial}_{tt} Y^n,\ \overline{\partial}_t Y^{n+1}) &= (\overline{\partial}_t W^n,\ W^{n+1}), \qquad W^n := \overline{\partial}_t Y^n,\ n = 0, 1, 2, \dots,\\ &= \frac{1}{2}\big[\,\overline{\partial}_t\|\,\overline{\partial}_t Y^n\|^2_{L^2} + \kappa\|\,\overline{\partial}_{tt} Y^n\|^2_{L^2}\big]. \end{aligned} \tag{3.5}$$

In addition one can verify that

$$
\begin{aligned}
(\overline{\partial}_{tt} Y^{n-1},\ Y^{n+1}) &= \kappa(\overline{\partial}_{tt} Y^{n-1},\ \overline{\partial}_t Y^n) \\
&\quad + \overline{\partial}_t(\overline{\partial}_t Y^{n-1},\ Y^n)\ - \|\ \overline{\partial}_t Y^n\|_{L^2}^2.
\end{aligned} \tag{3.6}
$$

Now we have the following lemma.

LEMMA 3.1. *If u_h^n solves* (3.1)*, then it satisfies*

$$
(\overline{\partial}_t u_h^n,\ \phi) - \sum_{i=1}^d (F_i(u_h^n),\ \partial_{x_i}\phi) + \varepsilon\left((\overline{\partial}_{tt} u_h^{n-1},\ \phi) + \sum_{i=1}^d (A_i \partial_{x_i} u_h^n,\ \partial_{x_i}\phi)\right) = 0. \tag{3.7}
$$

Proof. For $\phi \in S_k$, we see that the solution of (3.1) satisfies

$$
\begin{aligned}
\sum_{i=1}^d (\overline{\partial}_t v_{h,i}^{n-1},\ \partial_{x_i}\phi) &= \sum_{i=1}^d \left(\frac{v_{h,i}^n - v_{h,i}^{n-1}}{\kappa},\ \partial_{x_i}\phi \right) \\
&\overset{(3.1)}{=} \left(\frac{\overline{\partial}_t u_h^n - \overline{\partial}_t u_h^{n-1}}{\kappa},\ \phi \right) = (\overline{\partial}_{tt} u_h^{n-1},\ \phi).
\end{aligned}
$$

Next, summing $i = 1, \dots, d$, (3.1), and using that $\partial_{x_i}\phi \in V_{k-1}$, we get

$$
\begin{aligned}
0 &= \sum_{i=1}^d (v_{h,i}^n,\ \partial_{x_i}\phi) - \sum_{i=1}^d (F_i(u_h^n),\ \partial_{x_i}\phi) + \varepsilon \sum_{i=1}^d (\overline{\partial}_t v_{h,i}^{n-1} + A_i \partial_{x_i} u_h^n,\ \partial_{x_i}\phi) \\
&\overset{(3.1)}{=} (\overline{\partial}_t u_h^n,\ \phi) - \sum_{i=1}^d (F_i(u_h^n),\ \partial_{x_i}\phi) + \varepsilon \sum_{i=1}^d (\overline{\partial}_t v_{h,i}^{n-1} + A_i \partial_{x_i} u_h^n,\ \partial_{x_i}\phi)
\end{aligned} \tag{3.8}
$$

and the result follows. ∎

In the case $d = 1$, we have

$$
(\overline{\partial}_t v_h^n,\ \partial_x\phi) = (\overline{\partial}_{tt} u_h^n,\ \phi), \tag{3.9}
$$

$$
(\overline{\partial}_t u_h^n,\ \phi) - (F(u_h^n),\ \partial_x\phi) + \varepsilon\big((\overline{\partial}_{tt} u_h^{n-1},\ \phi) + (A\partial_x u_h^n,\ \partial_x\phi)\big) = 0. \tag{3.10}
$$

3.2. The case $d = 1$ and F' symmetric. Let $\phi = 2u_h^{n+1} + 4\varepsilon\ \overline{\partial}_t u_h^n,$ in (3.10). Then

$$
\begin{aligned}
0 &= 2(\overline{\partial}_t u_h^n,\ u_h^{n+1})\ + 2(\partial_x F(u_h^n),\ u_h^{n+1}) \\
&\quad + 2\varepsilon(\overline{\partial}_{tt} u_h^{n-1},\ u_h^{n+1})\ + 2\varepsilon(A\ \partial_x u_h^n,\ \partial_x u_h^{n+1}) \\
&\quad + 4\varepsilon(\overline{\partial}_t u_h^n,\ \overline{\partial}_t u_h^n)\ + 4\varepsilon(\partial_x F(u_h^n),\ \overline{\partial}_t u_h^n) \\
&\quad + 4\varepsilon^2(\overline{\partial}_{tt} u_h^{n-1},\ \overline{\partial}_t u_h^n)\ + 4\varepsilon^2(A\ \partial_x u_h^n,\ \partial_x \overline{\partial}_t u_h^n).
\end{aligned} \tag{3.11}
$$

Using the properties of the discrete time operators listed above, the terms of (3.11) are handled as follows. First note

$$
2(\overline{\partial}_t u_h^n,\ u_h^{n+1}) = \overline{\partial}_t \|\ u_h^n\|_{L^2}^2 + \kappa \|\ \overline{\partial}_t u_h^n\|_{L^2}^2.
$$

Also,

$$
2(\partial_x F(u_h^n),\ u_h^{n+1})\ = 2\ \kappa (F'(u_h^n)\ \partial_x u_h^n,\ \overline{\partial}_t u_h^n).
$$

The next term is estimated as

$$2\varepsilon(\overline{\partial}_{tt}u_h^{n-1},\ u_h^{n+1}) \overset{(3.6)}{=} 2\varepsilon\overline{\partial}_t(\overline{\partial}_t u_h^{n-1},\ u_h^n)\ - 2\varepsilon\|\,\overline{\partial}_t u_h^n\|_{L^2}^2\ + 2\varepsilon\kappa(\overline{\partial}_{tt}u_h^{n-1},\ \overline{\partial}_t u_h^n)$$
$$\geq 2\varepsilon\overline{\partial}_t(\overline{\partial}_t u_h^{n-1},\ u_h^n) - 2\varepsilon\|\,\overline{\partial}_t u_h^n\|_{L^2}^2 - 2\varepsilon^2\kappa\|\,\overline{\partial}_{tt}u_h^{n-1}\|_{L^2}^2 - \frac{\kappa}{2}\|\,\overline{\partial}_t u_h^n\|_{L^2}^2.$$

In addition,

$$2\varepsilon(A\,\partial_x u_h^n,\ \partial_x u_h^{n+1})\ = 2\varepsilon(A\,\partial_x u_h^n,\ \partial_x u_h^n)\ + 2\varepsilon\,\kappa(A\,\partial_x u_h^n,\ \partial_x\overline{\partial}_t u_h^n).$$

For the terms with coefficient 4ε we first note

$$4\varepsilon^2(\overline{\partial}_{tt}u_h^{n-1},\ \overline{\partial}_t u_h^n) \overset{(3.5)}{=} 2\varepsilon^2\ \overline{\partial}_t\|\,\overline{\partial}_t u_h^{n-1}\|_{L^2}^2 + 2\varepsilon^2\ \kappa\|\,\overline{\partial}_{tt}u_h^{n-1}\|_{L^2}^2$$

and

$$4\varepsilon^2(A\,\partial_x u_h^n,\ \partial_x\overline{\partial}_t u_h^n)\ = 4\varepsilon^2(W^n,\ \overline{\partial}_t W^n), \qquad W^n := A^{1/2}\partial_x u_h^n,\ n = 0, 1, 2, \dots,$$
$$\overset{(3.4)}{=} 2\varepsilon^2\ \overline{\partial}_t\|\,W^n\|_{L^2}^2 - 2\varepsilon^2\ \kappa\|\,\overline{\partial}_t W^n\|_{L^2}^2$$
$$= 2\varepsilon^2\ \overline{\partial}_t(A\,\partial_x u_h^n,\ \partial_x u_h^n)\ - 2\varepsilon^2\ \kappa(A\,\partial_x\overline{\partial}_t u_h^n,\ \partial_x\overline{\partial}_t u_h^n).$$

Summarizing, the terms with discrete time derivative that will appear in (3.11) are

$$\overline{\partial}_t\Big[\|\,u_h^n\|_{L^2}^2\ + 2\varepsilon(\overline{\partial}_t u_h^{n-1},\ u_h^n)\ + 2\varepsilon^2\|\,\overline{\partial}_t u_h^{n-1}\|_{L^2}^2\ + 2\varepsilon^2(A\,\partial_x u_h^n,\ \partial_x u_h^n)\Big]$$
$$= \overline{\partial}_t\Big[\|\,u_h^n\ + \varepsilon\,\overline{\partial}_t u_h^{n-1}\|_{L^2}^2\ + \varepsilon^2\|\,\overline{\partial}_t u_h^{n-1}\|_{L^2}^2\ + 2\varepsilon^2(A\,\partial_x u_h^n,\ \partial_x u_h^n)\Big].$$

In addition, the following calculation is useful:

$$2\varepsilon\|\,\overline{\partial}_t u_h^n\|_{L^2}^2 + 4\varepsilon(\partial_x F(u_h^n),\ \overline{\partial}_t u_h^n)$$
$$= \varepsilon\|\,\overline{\partial}_t u_h^n\|_{L^2}^2 + 2\varepsilon\|\ \frac{1}{\sqrt{2}}\,\overline{\partial}_t u_h^n + \sqrt{2}\ F'(u_h^n)\ \partial_x u_h^n\|_{L^2}^2$$
$$- 4\varepsilon((F'(u_h^n))^2\ \partial_x u_h^n,\ \partial_x u_h^n).$$

We conclude, therefore, that

$$\begin{aligned}&\overline{\partial}_t\Big[\ \|\,u_h^n\ + \varepsilon\,\overline{\partial}_t u_h^{n-1}\|_{L^2}^2\ + \varepsilon^2\|\,\overline{\partial}_t u_h^{n-1}\|_{L^2}^2\ + 2\varepsilon^2(A\,\partial_x u_h^n,\ \partial_x u_h^n)\Big]\\ &\quad + \varepsilon\|\,\overline{\partial}_t u_h^n\|_{L^2}^2 + \frac{\kappa}{2}\|\,\overline{\partial}_t u_h^n\|_{L^2}^2\ + 2\varepsilon((A - 2(F'(u_h^n))^2)\ \partial_x u_h^n,\ \partial_x u_h^n)\\ &\quad \leq |2\kappa(\partial_x F(u_h^n),\ \overline{\partial}_t u_h^n)| + |2\varepsilon\ \kappa(A\,\partial_x u_h^n,\ \partial_x\overline{\partial}_t u_h^n)|\\ &\quad + 2\varepsilon^2\ \kappa(A\,\partial_x\overline{\partial}_t u_h^n,\ \partial_x\overline{\partial}_t u_h^n).\end{aligned} \tag{3.12}$$

Next,

$$|2\kappa(\partial_x F(u_h^n),\ \overline{\partial}_t u_h^n)| \leq 4\kappa((F'(u_h^n))^2\ \partial_x u_h^n,\ \partial_x u_h^n) + \frac{\kappa}{4}\|\,\overline{\partial}_t u_h^n\|_{L^2}^2.$$

We will use the inverse inequality in S_k [7],

$$\|\partial_x\varphi\|_{L^2} \leq C_I\underline{h}^{-1}\|\varphi\|_{L^2} \qquad \forall\varphi \in S_k, \tag{3.13}$$

to obtain

$$|2\varepsilon\, \kappa(A\, \partial_x u_h^n,\ \partial_x \overline{\partial}_t u_h^n)| \le \varepsilon\, C_I\, \|A\|\, \frac{\kappa}{\underline{h}}\, \|\partial_x u_h^n\|_{L^2}^2 + \varepsilon\, C_I\, \|A\|\, \frac{\kappa}{\underline{h}}\, \|\overline{\partial}_t u_h^n\|_{L^2}^2,$$

$$2\varepsilon^2\, \kappa(A\, \partial_x \overline{\partial}_t u_h^n,\ \partial_x \overline{\partial}_t u_h^n) \le \varepsilon\, \frac{\varepsilon}{\underline{h}} \left(C_I^2\, \|A\|\, \frac{\kappa}{\underline{h}} \right) \|\overline{\partial}_t u_h^n\|_{L^2}^2.$$

Multiplying (3.12) by κ, and summing we finally conclude with the following proposition.

PROPOSITION 3.1. *We assume that $F'(u)$ is symmetric and that for given $\tilde{\beta}$ there holds*

$$\kappa \le \tilde{\beta}\varepsilon. \tag{3.14}$$

Assume further that we can choose A symmetric so that for some ν,

$$A - (\,2 + 4\tilde{\beta}\,)\, F'(u)^2 \ge \nu\, I \quad \textit{for} \quad u \in \mathbb{R}^n. \tag{3.15}$$

Let $\gamma_{CFL} = C_I^2\, \|A\| \frac{\kappa}{\underline{h}}$ and assume that γ_{CFL} is sufficiently small and that

$$\varepsilon \le \frac{1}{2\,\gamma_{CFL}}\, \underline{h}.$$

Then the approximations of the fully discrete schemes satisfy the stability estimate

$$\begin{aligned} &\|\, u_h^n + \varepsilon\, \overline{\partial}_t u_h^{n-1}\|_{L^2}^2 + \varepsilon^2 \|\, \overline{\partial}_t u_h^{n-1}\|_{L^2}^2 + 2\varepsilon^2 (A\, \partial_x u_h^n,\ \partial_x u_h^n) \\ &\quad + \sum_{j=1}^{n-1} \varepsilon\, \kappa \|\overline{\partial}_t u_h^j\|_{L^2}^2 + \sum_{j=1}^{n-1} \kappa^2 \|\, \overline{\partial}_t u_h^j\|_{L^2}^2 + \sum_{j=1}^{n-1} \varepsilon\, \kappa\, \|\partial_x u_h^j\|_{L^2}^2 \le C(u_h^0). \end{aligned}$$

In what follows we study the compactness properties of the dissipation measure associated to the scheme. To this end we use the notation

$$\begin{aligned} &u_h \quad \text{denotes the piecewise linear in time} \\ &\qquad \text{function such that} \quad u_h(t^n) = u_h^n, \\ &\overline{u}_h \quad \text{denotes the piecewise constant in time} \\ &\qquad \text{function such that} \quad \overline{u}_h(t^n) = u_h^n,\ I_n = (t^n, t^{n+1}]. \end{aligned} \tag{3.16}$$

PROPOSITION 3.2. *Under the assumptions of Proposition 3.1, for entropy pairs (η, q) such that*

$$\|\eta\|_{L^\infty},\ \|q\|_{L^\infty},\ \|\eta'\|_{L^\infty},\ \|\eta''\|_{L^\infty} \le C$$

and for $h \le C\,\varepsilon$ there holds

$$\eta(u_h)_t + q(u_h)_x \quad \textit{lies in a compact set of } H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+), \tag{3.17}$$

where u_h is defined by (3.16).

Proof. Let (η, q) be an entropy pair and $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$ a test function, and $\operatorname{supp} \phi \subset \tilde{\Omega} \times [0, \tilde{T}] =: Q$. Without loss of generality assume that $\tilde{T} = t^{m+1}$. Let

$\Pi : H^1 \to S_k$ be the L^2-projection onto the finite element space defined in (2.6). Then using the definition of the scheme we obtain

(3.18)

$$\begin{aligned}
\int_0^{\tilde T} \Big(\partial_t \eta(u_h) \; + \; \partial_x q(u_h), \phi\Big) dt \; &= \; \int_0^{\tilde T} \Big(\eta'(u_h)\big[\partial_t u_h \; + \; \partial_x F'(u_h) u_h\big], \phi\Big) dt \\
&= \; -\varepsilon \sum_{j=0}^{j=m} \int_{I_j} \Big\{ \Big(A\partial_x \; u_h^j, \partial_x \big[\Pi\big(\eta'(u_h)\phi\big)\big]\Big) + \Big(\overline{\partial}_{tt} u_h^{j-1}, \Pi\big(\eta'(u_h)\phi\big)\Big)\Big\} dt \\
&+ \sum_{j=0}^{j=m} \int_{I_j} \Big\{ \Big(\big[\overline{\partial}_t u_h^j \; + \; \partial_x F'(\overline{u}_h)\big], \; \eta'(u_h)\phi - \Pi\big(\eta'(u_h)\phi\big)\Big) \\
&+ \Big(\eta'(u_h)\partial_x\big[\, F(u_h) - F(\overline{u}_h)\big], \phi\Big)\Big\} dt.
\end{aligned}$$

Note here that for notational simplicity when we use $\overline{\partial}_t u_h^j, u_h^j, \overline{\partial}_{tt} u_h^{j-1}$ we mean the piecewise constant (with respect to t) functions that have these values in I_j. To proceed with the estimates, note that using (2.8) one obtains

(3.19)

$$\begin{aligned}
&\varepsilon \sum_{j=0}^{j=m} \int_{I_j} \Big| \; \Big(A\partial_x \; u_h^j, \partial_x \big[\Pi\big(\eta'(u_h)\phi\big)\big]\Big) \; \Big| dt \\
&\qquad \le \varepsilon \; C \left(\kappa \sum_{j=0}^{m} \|\partial_x \; u_h^j\|^2_{L^2(\Omega)} \right)^{1/2} \|\partial_x \; \big(\eta'(u_h)\phi\big)\|_{L^2(Q)} \\
&\qquad \le C \left(\varepsilon \, \kappa \sum_{j=0}^{m} \|\partial_x \; u_h^j\|^2(\Omega) \right) \cdot \|\eta''\|_{L^\infty} \; \|\phi\|_{C^0(Q)} \\
&\qquad\quad + \varepsilon^{1/2} \; C \left(\varepsilon \, \kappa \sum_{j=0}^{m} \|\partial_x \; u_h^j\|^2(\Omega) \right)^{1/2} \|\eta\|_{L^\infty} \; \|\partial_x \phi\|_{L^2(Q)}.
\end{aligned}$$

In addition, using the notation

$$\overline{v}_j = \kappa^{-1} \int_{I_j} v \; dt,$$

we have by (2.6),

(3.20)

$$\begin{aligned}
&- \varepsilon \sum_{j=0}^{m} \int_{I_j} \int_{\Omega} \overline{\partial}_{tt} u_h^{j-1} \; \Pi\Big(\eta'(u_h)\phi\Big) \; = \; -\varepsilon \sum_{j=0}^{m} \int_{I_j} \overline{\partial}_{tt} u_h^{j-1} \int_{\Omega} \eta'(u_h)\phi \\
&\quad = -\varepsilon \int_{\Omega} \sum_{j=0}^{m} (\overline{\partial}_t u_h^j - \overline{\partial}_t u_h^{j-1}) \; \overline{(\eta'(u_h)\phi)}_j \\
&\quad = \varepsilon \int_{\Omega} \sum_{j=0}^{m-1} \; \overline{\partial}_t u_h^j \; (\; \overline{(\eta'(u_h)\phi)}_{j+1} - \overline{(\eta'(u_h)\phi)}_j \;) - \varepsilon \int_{\Omega} \; \overline{\partial}_t u_h^m \int_{\Omega} \; \overline{(\eta'(u_h)\phi)}_m.
\end{aligned}$$

The stability in Proposition 3.1 implies, since $|\overline{v}_j| \le \| \, v \, \|_\infty$,

$$\varepsilon \Big| \int_\Omega \, \overline{\partial}_t u_h^m \, \overline{(\eta'(u_h)\phi)}_m \Big| \; \le \; \varepsilon \, \|\overline{\partial}_t u_h^m\|_{L^2(\Omega)} \, \|\eta'\|_{L^\infty} \, \|\phi\|_{C^0(\Omega)} \, m(\Omega)^{1/2} \le \; C_\Omega \|\phi\|_{C^0(Q)}. \tag{3.21}$$

Observing that $|\overline{v}_{j+1} - \overline{v}_j| = \frac{1}{\kappa} \left| \int_{I_j} \int_t^{t+\kappa} v_t \, ds \, dt \right| \le \int_{t_j}^{t_{j+2}} | \, v_t \, | \, dt$, we conclude

$$\begin{aligned} &\varepsilon \left| \int_\Omega \sum_{j=0}^{m-1} \overline{\partial}_t u_h^j \, (\overline{((\eta'(u_h)\phi)}_{j+1} - \overline{(\eta'(u_h)\phi)}_j) \right| \\ &\quad \le C \left(\varepsilon \, \kappa \sum_{j=0}^{m} \|\overline{\partial}_t \, u_h^j\|^2_{L^2(\Omega)} \right) \|\eta''\|_{L^\infty} \, \|\phi\|_{C^0(Q)} \\ &\quad + \varepsilon^{1/2} \left(\varepsilon \, \kappa \sum_{j=0}^{m} \|\overline{\partial}_t \, u_h^j\|^2_{L^2(\Omega)} \right)^{1/2} \|\eta'\|_{L^\infty} \, \|\partial_t \phi\|_{L^2(Q)}. \end{aligned} \tag{3.22}$$

Next,

$$\begin{aligned} &\|\eta'(u_h)\phi - \Pi\big(\eta'(u_h)\phi\big)\|_{L^2(\Omega)} \\ &\quad \le Ch\|\eta''\|_{L^\infty} \, \|\partial_x u_h\|_{L^2(\Omega)} \, \|\phi\|_{C^0(\Omega)} + Ch\|\eta'\|_{L^\infty} \, \|\partial_x \phi\|_{L^2(\Omega)} \end{aligned}$$

and $\|F'(u)^2\|_{L^\infty} \le C$ (see (2.4)); therefore,

$$\begin{aligned} &\sum_{j=0}^{m} \int_{I_j} \left| \Big(\big[\, \overline{\partial}_t u_h^j \; + \; F'(\overline{u}_h)\partial_x \overline{u}_h \big], \; \eta'(u_h)\phi - \Pi\big(\eta'(u_h)\phi\big) \Big) \right| \\ &\quad \le C \left(h \, \kappa \sum_{j=0}^{m} \|\overline{\partial}_t \, u_h^j\|^2_{L^2(\Omega)} + \|\partial_x \, u_h^j\|^2_{L^2(\Omega)} \right) \|\phi\|_{C^0(Q)} \\ &\quad + h \left(\kappa \sum_{j=0}^{m} \|\overline{\partial}_t \, u_h^j\|^2_{L^2(\Omega)} + \|\partial_x \, u_h^j\|^2_{L^2(\Omega)} \right)^{1/2} \|\partial_x \phi\|_{L^2(Q)}. \end{aligned} \tag{3.23}$$

Finally, using the fact that $|u_h - \overline{u}_h| \le C\kappa |\partial_t u_h| = C\kappa |\overline{\partial}_t \, u_h^n|$, we have by using (3.14) and (3.15),

$$\begin{aligned} &\sum_{j=0}^{m} \int_{I_j} \Big(\eta'(u_h)\partial_x \big[\, F(u_h) - F(\overline{u}_h) \big], \phi \Big) = - \sum_{j=0}^{m} \int_{I_j} \Big(\big[\, F(u_h) - F(\overline{u}_h) \big], \partial_x(\eta'(u_h) \, \phi) \Big) \\ &\quad \le C \left(\varepsilon \, \kappa \sum_{j=0}^{m} \|\overline{\partial}_t \, u_h^j\|^2_{L^2(\Omega)} + \|\partial_x \, u_h^j\|^2_{L^2(\Omega)} \right) \|\phi\|_{C^0(Q)} \\ &\quad + \varepsilon \left(\kappa \sum_{j=0}^{m} \|\overline{\partial}_t \, u_h^j\|^2_{L^2(\Omega)} + \|\partial_x \, u_h^j\|^2_{L^2(\Omega)} \right)^{1/2} \|\partial_x \phi\|_{L^2(Q)}. \end{aligned} \tag{3.24}$$

Combining (3.19)–(3.24), we obtain the desired result in view of Lemma 2.2. ∎

3.3. The case $d = 1$, and the system admits a convex entropy. The case that F' is not necessarily symmetric but the system is equipped with a convex entropy η will be briefly examined here. The analysis in this case mainly uses a combination of arguments from the corresponding semidiscrete case and the analysis of the fully discrete scheme in the symmetric case. For this reason we will present briefly the basic steps of the proof, explaining only the new estimates. The following proposition holds.

PROPOSITION 3.3. *Assume that* (1.9) *admits a convex entropy* $\eta(u)$ *satisfying* (2.13)*, and the symmetric, positive definite matrix* A *satisfies* (2.14) *for some* $\nu > 0$ *where the constant* $\overline{\alpha}$ *depends on* $\alpha, \beta,$ *and* $\tilde{\beta}$*; see* (2.13), (2.22), *and* (3.14). *Under similar conditions on* κ, ε, h *as in Proposition* 3.1 *(with possibly different constants), and if* $h \leq \gamma\, \varepsilon$ *for some* $\gamma > 0$*, the fully discrete finite element approximations satisfy*

$$\begin{aligned} &\| u_h^n + \varepsilon\, \overline{\partial}_t u_h^{n-1}\|_{L^2}^2 + \varepsilon^2 \| \overline{\partial}_t u_h^{n-1}\|_{L^2}^2 + 2\varepsilon^2 (A\, \partial_x u_h^n,\ \partial_x u_h^n) \\ &\quad + \sum_{j=1}^{n-1} \varepsilon\, \kappa \| \overline{\partial}_t u_h^j\|_{L^2}^2 + \sum_{j=1}^{n-1} \kappa^2 \| \overline{\partial}_t u_h^j\|_{L^2}^2 + \sum_{j=1}^{n-1} \varepsilon\, \kappa\, \|\partial_x u_h^j\|_{L^2}^2 \leq C(u_h^0). \end{aligned} \tag{3.25}$$

Proof. The fully discrete finite element approximation u_h^n satisfies

$$(\overline{\partial}_t u_h^n,\ \phi) - (F(u_h^n),\ \partial_x \phi) + \varepsilon\big((\overline{\partial}_{tt} u_h^{n-1},\ \phi) + (A \partial_x u_h^n,\ \partial_x \phi)\big) = 0. \tag{3.26}$$

Let $\phi = P_1\, \eta'(u_h^{n+1})$ in (3.26), where $P_1 : H^1 \to S_k$ is the elliptic projection defined in (2.20). Then

$$\begin{aligned} &(\overline{\partial}_t u_h^n, \eta'(u_h^{n+1})) + \big(\partial_x F(u_h^n), \eta'(u_h^{n+1})\big) \\ &\quad + \varepsilon(\overline{\partial}_{tt} u_h^{n-1}, P_1, \eta'(u_h^{n+1})) + \varepsilon(A \partial_x\, u_h^n, \partial_x \eta'(u_h^{n+1})) \\ &\quad = (\overline{\partial}_t u_h^n, \eta'(u_h^{n+1}) - P_1\, \eta'(u_h^{n+1})) \\ &\quad + \big(\partial_x F(u_h^n), \eta'(u_h^{n+1}) - P_1\, \eta'(u_h^{n+1})\big) =: Z_1 + Z_2. \end{aligned} \tag{3.27}$$

The terms in the right-hand side will be estimated as in the semidiscrete case. We start by examining the stability that is inherited in the left-hand side. In a way similar to (3.6) one can show

$$\begin{aligned} (\overline{\partial}_{tt} Y^{n-1},\ W^{n+1}) &= \kappa(\overline{\partial}_{tt} Y^{n-1},\ \overline{\partial}_t W^n) \\ &\quad + \overline{\partial}_t(\overline{\partial}_t Y^{n-1},\ W^n) - (\overline{\partial}_t Y^n,\ \overline{\partial}_t W^n). \end{aligned} \tag{3.28}$$

Therefore,

$$\begin{aligned} &\varepsilon(\overline{\partial}_{tt} u_h^{n-1}, P_1\, \eta'(u_h^{n+1})) \\ &\quad = \varepsilon\kappa(\overline{\partial}_{tt} u_h^{n-1},\ \overline{\partial}_t P_1\, \eta'(u_h^n)) + \varepsilon\overline{\partial}_t(\overline{\partial}_t u_h^{n-1},\ P_1\, \eta'(u_h^n)) - \varepsilon(\overline{\partial}_t u_h^n,\ \overline{\partial}_t P_1\, \eta'(u_h^n)) \\ &\quad = \varepsilon\overline{\partial}_t(\overline{\partial}_t u_h^{n-1},\ \eta'(u_h^n)) + \varepsilon\kappa(\overline{\partial}_{tt} u_h^{n-1},\ \overline{\partial}_t P_1\, \eta'(u_h^n)) \\ &\qquad - \varepsilon(\overline{\partial}_t u_h^n,\ \overline{\partial}_t P_1\, \eta'(u_h^n)) + \varepsilon\overline{\partial}_t(\overline{\partial}_t u_h^{n-1},\ P_1\, \eta'(u_h^n) - \eta'(u_h^n)). \end{aligned} \tag{3.29}$$

Taylor's formula implies

$$\begin{aligned} \int_\Omega \eta(u_h^n) dx &= \int_\Omega \eta(u_h^{n+1}) dx - \kappa(\eta'(u_h^{n+1}), \overline{\partial}_t u_h^n) \\ &\quad + \kappa^2 \bigg(\overline{\partial}_t u_h^n, \bigg\{\int_0^1 \int_0^s \eta''(u_h^{n+1} - \kappa\, \tau \overline{\partial}_t u_h^n) d\tau ds\bigg\} \overline{\partial}_t u_h^n\bigg), \end{aligned} \tag{3.30}$$

i.e.,

$$
\begin{aligned}
(\overline{\partial}_t u_h^n, \eta'(u_h^{n+1})) = {} & \overline{\partial}_t \int_\Omega \eta(u_h^n) dx \\
& + \kappa \Big(\overline{\partial}_t u_h^n, \Big\{ \int_0^1 \int_0^s \eta''(u_h^{n+1} - \kappa\,\tau \overline{\partial}_t u_h^n) d\tau ds \Big\} \overline{\partial}_t u_h^n \Big).
\end{aligned} \tag{3.31}
$$

Further, since (η, q) is an entropy pair,

$$
\begin{aligned}
(F'(u_h^n)\,\partial_x u_h^n,\ \eta'(u_h^{n+1})) &= (F'(u_h^n)\,\partial_x u_h^n,\ \eta'(u_h^n)) \\
&\quad + (F'(u_h^n)\,\partial_x u_h^n,\ \eta'(u_h^{n+1}) - \eta'(u_h^n)) \\
&= \kappa (F'(u_h^n)\,\partial_x u_h^n,\ \overline{\partial}_t \eta'(u_h^n)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\overline{\partial}_t \int_\Omega & \eta(u_h^n) dx + \varepsilon \overline{\partial}_t (\overline{\partial}_t u_h^{n-1},\ \eta'(u_h^n)) \\
& + \varepsilon (A \partial_x\, u_h^n, \eta''(u_h) \partial_x\, u_h^n) - \varepsilon (\overline{\partial}_t\, u_{h^n}, P_1\, \overline{\partial}_t \eta'(u_h^n)) \\
& + \kappa \Big(\overline{\partial}_t u_h^n, \Big\{ \int_0^1 \int_0^s \eta''(u_h^{n+1} - \kappa\,\tau \overline{\partial}_t u_h^n) d\tau ds \Big\} \overline{\partial}_t u_h^n \Big) \\
& = Z_1 + Z_2 + Z_3,
\end{aligned} \tag{3.32}
$$

where the new term Z_3 is given by

$$
\begin{aligned}
Z_3 = & - \varepsilon \kappa (\overline{\partial}_{tt} u_h^{n-1},\ \overline{\partial}_t P_1\, \eta'(u_h^n)) \\
& - \varepsilon \overline{\partial}_t (\overline{\partial}_t u_h^{n-1},\ P_1\, \eta'(u_h^n) - \eta'(u_h^n)) - \kappa (F'(u_h^n)\,\partial_x u_h^n,\ \overline{\partial}_t \eta'(u_h^n)).
\end{aligned} \tag{3.33}
$$

Using once more Taylor's formula we obtain,

$$
\begin{aligned}
\int_\Omega \eta(u_h^n + \varepsilon \overline{\partial}_t u_h^{n-1}) dx = {} & \int_\Omega \eta(u_h^n) dx + \varepsilon \overline{\partial}_t (\overline{\partial}_t u_h^{n-1},\ \eta'(u_h^n)) \\
& + \varepsilon^2 \Big(\overline{\partial}_t u_h^{n-1}, \Big\{ \int_0^1 \int_0^s \eta''(u_h^n + \varepsilon\,\tau \overline{\partial}_t u_h^{n-1}) d\tau ds \Big\} \overline{\partial}_t u_h^{n-1} \Big).
\end{aligned} \tag{3.34}
$$

By a slight modification of the proof of Lemma 2.3 we have

$$
\varepsilon |(\overline{\partial}_t\, u_h^n, P_1\, \overline{\partial}_t \eta'(u_h^n))| \le \beta\, \|\eta''\|_{L^\infty}\, \|\overline{\partial}_t\, u_h^n\|^2_{L^2(\Omega)}. \tag{3.35}
$$

Essentially what remains now is an estimate of $\|\overline{\partial}_t\ u_h^n\|_{L^2(\Omega)}$. As in the symmetric case we use the test function $\phi = \overline{\partial}_t\ u_h^n$ and we conclude the proof by combining arguments from the semidiscrete case (see (2.33)–(2.37)), and the fully discrete case with symmetric F' (cf. the terms with coefficient $4\varepsilon^2$), and by estimating of course the terms Z_i. It is to be noted, finally, the essential role of the estimate

$$
\begin{aligned}
\kappa \Big(\overline{\partial}_t u_h^n, & \Big\{ \int_0^1 \int_0^s \eta''(u_h^{n+1} - \kappa\,\tau \overline{\partial}_t u_h^n) d\tau ds \Big\} \overline{\partial}_t u_h^n \Big) \\
& \ge \mu \kappa \|\overline{\partial}_t u_h^n\|^2_{L^2}, \qquad \mu > 0,
\end{aligned} \tag{3.36}
$$

in the stability analysis. ∎

Remark 3.1 (mesh conditions). Proposition 3.3 holds under the assumptions for the mesh stated in Proposition 3.1, assuming in addition that $h \le \gamma\varepsilon$. Combining these conditions we conclude that we need to have a CFL condition with small constant γ_{CFL} and in addition $h \le \frac{\gamma}{2\gamma_{CFL}}\underline{h}$. This last relation is a quasi-uniformity condition on the mesh, the constant of which depends on how strong the CFL condition is. It seems that it is a weakness of our proof to assume $h \le \gamma\varepsilon$ rather than $h_{\text{loc}} \le \gamma\varepsilon$, where h_{loc} is the local mesh size close to the shock; see section 1.2. If this were the case this would not be a restriction since h_{loc} is naturally of the order of $\underline{h}$. Nevertheless, the above conditions provide enough room for computations compatible with the principle to have finer mesh in the shock areas and coarser mesh in the smooth parts of the solution. See also the related discussion in section 6.

We conclude with the following proposition.

PROPOSITION 3.4. *For entropy pairs* (η, q) *such that*

$$\|\eta\|_{L^\infty},\ \|q\|_{L^\infty},\ \|\eta'\|_{L^\infty},\ \|\eta''\|_{L^\infty}\ \le\ C$$

and under the hypotheses of Proposition 3.3*, we have*

$$\eta(u_h)_t\ +\ q(u_h)_x\ \subset\ \textit{lies in a compact set of } H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+),$$

where u_h *and* u_h^n *are related by* (3.16)*.*

3.4. Estimates in the multidimensional case. Let (1.1) be endowed with a uniformly convex entropy η; the fluxes q_i are given by (2.42) [14, sec. IV.4.3]. The finite element approximations defined by (3.1) satisfy similar a priori bounds with the one-dimensional case. The matrices A_i should now satisfy the analogue of (2.44). We state the stability estimate; its proof is a modification of the proof of Proposition 3.3 and is omitted.

PROPOSITION 3.5. *Assume that* (1.1) *is equipped with a convex entropy* $\eta(u)$ *satisfying* (2.6)*. If the symmetric, positive definite matrices* A_i *satisfy* (2.44)*, then, under similar conditions on* κ, ε, h *as in Proposition* 3.1 *(with possibly different constants), and for* $h \le \gamma\varepsilon$ *for some* $\gamma > 0$*, the fully discrete finite element approximations* (3.1) *satisfy*

$$\begin{aligned}\|\, u_h^n\ + \varepsilon\, \overline{\partial}_t u_h^{n-1}\|^2_{L^2}\ + \varepsilon^2 \|\,\overline{\partial}_t u_h^{n-1}\|^2_{L^2}\ + 2\varepsilon^2 \sum_{i=1}^{d} (A_i\, \partial_{x_i} u_h^n,\ \partial_{x_i} u_h^n) \\ + \sum_{j=1}^{n-1} \varepsilon\, \kappa \|\,\overline{\partial}_t u_h^j\|^2_{L^2} + \sum_{j=1}^{n-1} \kappa^2 \|\,\overline{\partial}_t u_h^j\|^2_{L^2} + \sum_{j=1}^{n-1} \varepsilon\, \kappa\ \sum_{i=1}^{d} \|\partial_{x_i} u_h^j\|^2_{L^2}\ \le\ C(u_h^0).\end{aligned}$$

4. Convergence of finite element schemes for one-dimensional systems. The compactness of the dissipation measure (2.41) or (3.17) is central in establishing compactness of approximate solutions for systems of conservation laws via the program of compensated compactness. Such results are available (in a one-dimensional context) for the scalar conservation law, the equations of elastodynamics, the equations of isentropic gas dynamics, and the class of rich systems (see [42, 15] and the references in [14, Chap. XV]). One difficulty in applying the compensated compactness framework is that, while several of the existing compactness theorems are valid in the presence of uniform L^∞-estimates, the available estimates in applications are often just in the energy norm. In particular, this is the case for the approximations arising via semidiscrete (2.1) or fully discrete (3.2) finite element schemes. Note that under the additional hypothesis of uniform L^∞ bounds for the approximations, one would conclude directly convergence toward a weak solution for all the aforementioned systems.

Our results can be applied to systems where the compensated compactness program has been carried out in the energy-norm framework. Such results are available for the scalar conservation law in the L^p framework (e.g., [35], [34, Thm. 2.3]) and for the equations of one-dimensional elasticity,

$$\begin{aligned} u_{1,t} - u_{2,x} &= 0, \\ u_{2,t} - \sigma(u_1)_x &= 0, \end{aligned} \tag{4.1}$$

in the energy norm [31, 38, 37]. In both cases one can deduce compactness of semidiscrete or fully discrete finite element schemes and conclude with a convergence result.

We consider here as a paradigm the system (4.1). For $\sigma'(u_1) > 0$, it is strictly hyperbolic with wave speeds $\lambda_{1,2} = \pm\sqrt{\sigma'(u_1)}$. It admits an infinite number of entropy pairs, of which the special pair

$$\eta = \frac{1}{2}u_2^2 + \int_0^{u_1} \sigma(\tau)d\tau, \quad q = -u_2\sigma(u_1) \tag{4.2}$$

is associated with the mechanical energy and the work of contact forces, and η is strictly convex. We assume that σ satisfies the subcharacteristic condition

$$0 < s \le \sigma'(u) \le S, \quad u \in \mathbb{R}, \tag{4.3}$$

with s, S positive constants. One easily checks that the matrix A can be selected so that all conditions in Propositions 2.3 and 3.3 hold.

We need a second hypothesis on σ that allows us to apply the results of [38, 37]. We assume either that (4.1) is genuinely nonlinear with

$$\sigma''(u) \neq 0 \quad \text{and} \quad \sigma'', \sigma''' \in L^2 \cap L^\infty(\mathbb{R}) \tag{4.4}$$

or that σ has precisely one inflection point at u_0 with

$$\begin{aligned} (u-u_0)\sigma''(u) \neq 0 \quad \text{for } u \neq u_0 \\ \text{and} \;\; \sigma'', \sigma''' \in L^2 \cap L^\infty(\mathbb{R}). \end{aligned} \tag{4.5}$$

We then have the following theorem.

THEOREM 4.1. *Let $\sigma \in C^3$ satisfy hypotheses* (4.3), (4.4) *(or* (4.3), (4.5)*). Let* $(u_1^{\varepsilon,h}, u_2^{\varepsilon,h})$ *be a family of solutions of* (2.1), *and let A be a symmetric, positive definite matrix satisfying* (2.14). *Then, for $h \le \gamma\varepsilon$ (with γ as in Proposition* 2.3*) and along a subsequence,*

$$u_{1,h} \to u_1\,, \quad u_{2,h} \to u_2\,, \quad a.e.\ (x,t) \text{ and in } L^p_{loc}(\mathbb{R}\times(0,T)) \text{ for } p<2,$$

and (u_1, u_2) *is a weak solution of* (4.1).

Proof. The proof uses the theory of compensated compactness and proceeds by controlling the dissipation measure

$$\partial_t \eta(u_1^{\varepsilon,h}, u_2^{\varepsilon,h}) + \partial_x q(u_1^{\varepsilon,h}, u_2^{\varepsilon,h}) \quad \text{lies in a compact of } H^{-1}_{loc}, \tag{4.6}$$

for entropy pairs $(\eta(u,v)\,,\, q(u,v))$ for the equations of elasticity. In the presence of uniform L^∞-bounds, the theorem of DiPerna [15] would guarantee compactness of approximate solutions and imply that, along a subsequence, $u_1^{\varepsilon,h} \to u_1$ and $u_2^{\varepsilon,h} \to u_2$ a.e. (x,t).

In the current case, uniform L^∞-estimates are not available and the natural stability framework is in the energy norm (see Proposition 2.3). Nevertheless, under hypothesis (4.3) and by Proposition 2.5, the dissipation measure is controlled for a class of entropy-flux pairs $(\eta(u,v)\,,\, q(u,v))$ satisfying the growth restrictions

$$\eta,\ q,\ \eta_u,\ \eta_v,\ \eta_{uu},\ \eta_{uv},\ \eta_{vv} \in L^\infty(\mathbb{R}^2). \tag{4.7}$$

This class of entropy pairs contains sufficient test pairs in order to achieve the reduction of the generalized Young measure to a point mass and to show strong convergence in L^p_{loc} for $p < 2$. The hypotheses (4.3)–(4.4) allow us to apply the result of Shearer [38], where the reduction is performed for the genuine nonlinear case, while the hypotheses (4.3)–(4.5) allow us to apply the corresponding reduction in Serre and Shearer [37] applicable to the case of elasticity with one inflection point. ∎

In a similar manner we can prove convergence of fully discrete finite element approximations (3.2) for the equations (4.1).

THEOREM 4.2. *Let σ be as in Theorem* 4.1 *and let A satisfy the hypotheses of Proposition* 3.3. *Let $(u_{1,h}, u_{2,h})$ be the fully discrete finite element approximations defined in* (3.16). *If the parameters κ, h, and ε are restricted by* (3.14) *and $h \le \gamma\varepsilon$ for some $\gamma > 0$, then along a subsequence*

$$u_{1,h} \to u_1, \quad u_{2,h} \to u_2, \quad \text{a.e. } (x,t) \text{ and in } L^p_{loc}(\mathbb{R}\times(0,T)) \text{ for } p<2,$$

and (u,v) is a weak solution of (4.1).

5. Error estimates for smooth solutions. In this section we consider the system of conservation laws

$$\partial_t u + \partial_x F(u) = 0 \tag{5.1}$$

and assume that (5.1) is endowed with a convex entropy $\eta(u)$. We let u be a classical solution of (5.1) defined on a maximal interval of existence and let U_ε be the smooth solution of the relaxation approximation

$$\partial_t U_\varepsilon + \partial_x F(U_\varepsilon) = \varepsilon A \partial_{xx} U_\varepsilon - \varepsilon \partial_{tt} U_\varepsilon\,. \tag{5.2}$$

We show

$$\|U_\varepsilon(t) - u(t)\|_{L^2} \le C(t,u)\,\varepsilon\,, \tag{5.3}$$

where the constant $C(t,u)$ depends on a strong norm of u and blows up at the critical time.

5.1. Motivation. It was established in Theorem 5.2.1 of [14] that the classical solution of (1.1) is unique among the class of admissible weak solutions in the case where the system admits a convex entropy. The result follows by showing a stability estimate in L^2:

$$\|u(t) - w(t)\|_{L^2} \le C(t,u)\,\|u(0) - w(0)\|_{L^2}. \tag{5.4}$$

Here u is the classical and w an admissible weak solution of (1.1). The main idea of the proof is to control the spatial integral of the quadratic in the $u-w$ function

$$H(u,w) = \eta(w) - \eta(u) - \eta'(u)(w-u). \tag{5.5}$$

This is made possible by the observation that certain quantities arising in the proof vanish when u is a classical solution and thus satisfies the entropy inequality as equality. Our idea is to use a similar approach to show the error estimate (5.3). A difficulty arises (except for handling the error terms in an appropriate way) that it is no longer possible to work with the same function H as in (5.5). On the other hand, the estimates in [43] and in section 2 suggest that when the system admits a convex entropy, we are able to control the quantity

$$\int \eta(U_\varepsilon + \varepsilon\partial_t U_\varepsilon)dx.$$

Motivated by these considerations, we introduce the functions

$$H_R(u, U_\varepsilon) = \eta\big(U_\varepsilon + \varepsilon\partial_t(U_\varepsilon - u)\big) - \eta(u) - \eta'(u)\big(U_\varepsilon - u + \varepsilon\partial_t(U_\varepsilon - u)\big), \tag{5.6}$$

$$Q\,(u, U_\varepsilon) = q(U_\varepsilon) \;-\; q(u) - \eta'(u)\;(F(U_\varepsilon) - F(u)). \tag{5.7}$$

The function H_R is the relaxational correction of (5.5) and is of quadratic order in the quantity $\big(U_\varepsilon - u + \varepsilon\partial_t(U_\varepsilon - u)\big)$. Control of $\|u(t) - U_\varepsilon(t)\|^2_{L^2}$ is achieved through the additional control of $\varepsilon^2\|\partial_t\,(U_\varepsilon - u)\|^2_{L^2}$ that is obtained from a separate estimate natural for approximations by wave equation (5.2).

5.2. The decay functional. The first objective is to establish that H_R is a Lyapunov functional. We begin with the derivation of the main decay identity.

Let η be the convex entropy with q the corresponding flux. The classical solution u satisfies

$$\partial_t\eta(u) + \partial_x q(u) = 0.$$

The approximate solution of (5.2) will henceforth be denoted by $U \equiv U_\varepsilon$. It satisfies the identities

$$\begin{aligned}
&\partial_t(U - u) + \partial_x(F(U) - F(u)) = \varepsilon A U_{xx} - \varepsilon U_{tt},\\
&\partial_t\eta'(u)(U - u) + \partial_x\eta'(u)(F(U) - F(u))\\
&\quad = \eta''(u)u_x \cdot [F(U) - F(u) - F'(u)(U - u)] + \varepsilon\eta'(u)\cdot AU_{xx} - \varepsilon\eta'(u)\cdot U_{tt},
\end{aligned}$$

where we use (5.1) and the fact that η is an entropy if and only if $(\eta''F')^T = \eta''F'$; see (2.42). Combining the above, we deduce

$$\begin{aligned}
&\partial_t[\eta(U) - \eta(u) - \eta'(u)(U - u)] + \partial_x[q(U) - q(u) - \eta'(u)(F(U) - F(u))]\\
&\qquad = -\eta''(u)u_x\cdot[F(U) - F(u) - F'(u)(U - u)]\\
&\qquad\quad + \varepsilon\big(\eta'(U) - \eta'(u)\big)\cdot AU_{xx} - \varepsilon\big(\eta'(U) - \eta'(u)\big)\cdot U_{tt}.
\end{aligned} \tag{5.8}$$

We now use (5.8) in conjunction with the identities

$$\begin{aligned}
(\eta'(U) - \eta'(u))\cdot U_{tt} &= \partial_t[(\eta'(U) - \eta'(u))\cdot(U_t - u_t)] - \eta''(U)(U_t - u_t)\cdot(U_t - u_t)\\
&\quad - (\eta''(U) - \eta''(u))u_t\cdot(U_t - u_t) + (\eta'(U) - \eta'(u))\cdot u_{tt},\\
(\eta'(U) - \eta'(u))\cdot AU_{xx} &= \partial_x[(\eta'(U) - \eta'(u))\cdot A(U - u)_x]\\
&\quad - \eta''(U)(U - u)_x\cdot A(U - u)_x\\
&\quad - (\eta''(U) - \eta''(u))u_x\cdot A(U - u)_x + (\eta'(U) - \eta'(u))\cdot Au_{xx}
\end{aligned}$$

and

$$\eta\big(U + \varepsilon\partial_t(U-u)\big) = \eta(U) + \eta'(U)\varepsilon\partial_t(U-u) + \varepsilon^2\partial_t(U-u)\cdot\overline{\eta''}\partial_t(U-u)$$
$$\text{with}\quad \overline{\eta''} = \int_0^1\int_0^s \eta''(U+\varepsilon\,\tau\partial_t\,(U-u))d\tau ds$$

to conclude

$$\begin{aligned}(5.9)\quad &\partial_t\{\eta(U+\varepsilon\partial_t(U-u)) - \eta(u) - \eta'(u)[U-u+\varepsilon\partial_t(U-u)]\\ &\qquad\qquad -\varepsilon^2\partial_t(U-u)\cdot\overline{\eta''}\partial_t(U-u)\}\\ &\quad +\partial_x\{q(U)-q(u)-\eta'(u)(F(U)-F(u))\}\\ &\quad +\varepsilon\{\eta''(U)(U-u)_x\cdot A(U-u)_x - \eta''(U)(U-u)_t\cdot(U-u)_t\}\\ &= \partial_x\{\varepsilon(\eta'(U)-\eta'(u))\cdot A(U-u)_x\} - \eta''(u)u_x\cdot[F(U)-F(u)-F'(u)(U-u)]\\ &\quad + a_{1t}+a_{2t}+b_{1x}+b_{2x}.\end{aligned}$$

The error terms a_{1t}, a_{2t}, b_{1x}, and b_{2x} are defined by

$$(5.10)\qquad \begin{aligned} a_{1t} &= \varepsilon\big(\eta''(U)-\eta''(u)\big)u_t\cdot(U_t-u_t),\\ a_{2t} &= -\varepsilon\big(\eta'(U)-\eta'(u)\big)\cdot u_{tt},\\ b_{1x} &= -\varepsilon\big(\eta''(U)-\eta''(u)\big)u_x\cdot A(U-u)_x,\\ b_{2x} &= \varepsilon\big(\eta'(U)-\eta'(u)\big)\cdot Au_{xx}\end{aligned}$$

and will be estimated in what follows.

Identity (5.9) is supplemented by a correction accounting for the fact that the third term is indefinite. The correcting identity is obtained by multiplying the equation

$$(U-u)_t + F'(U)(U-u)_x = \varepsilon A(U-u)_{xx} - \varepsilon(U-u)_{tt} + \varepsilon(Au_{xx}-u_{tt}) - (F'(U)-F'(u))u_x$$

by $(U-u)_t$ and integrating by parts to deduce

$$(5.11)\qquad \begin{aligned}&\partial_t\Big\{\frac{1}{2}\varepsilon|U_t-u_t|^2 + \frac{1}{2}\varepsilon(U-u)_x\cdot A(U-u)_x\Big\} + |(U-u)_t|^2\\ &\quad + F'(U)(U-u)_x\cdot(U-u)_t = \partial_x\Big\{\varepsilon A(U-u)_x\cdot(U-u)_t\Big\} + c_{1t} + c_{2t},\end{aligned}$$

where c_{1t}, c_{2t} are given by

$$(5.12)\qquad \begin{aligned} c_{1t} &= \varepsilon(Au_{xx}-u_{tt})\cdot(U-u)_t,\\ c_{2t} &= -(F'(U)-F'(u))u_x\cdot(U-u)_t.\end{aligned}$$

Next, we multiply (5.11) by $2\alpha\varepsilon$, add the resulting identity to (5.9), and use (5.6) and (5.7) to arrive at

$$
\begin{aligned}
&\partial_t \mathcal{G}(u,U) + \partial_x Q(u,U) + \alpha\varepsilon \Big| (U-u)_t + F'(U)(U-u)_x \Big|^2 \\
&\quad + \varepsilon \Big\{ \eta''(U)(U-u)_x \cdot A(U-u)_x - \alpha F'(U)(U-u)_x \cdot F'(U)(U-u)_x \Big\} \\
&\quad + \varepsilon \Big\{ \big(\alpha I - \eta''(U)\big)(U-u)_t \cdot (U-u)_t \Big\} \\
&= \partial_x \Big\{ \varepsilon\big(\eta'(U) - \eta'(u)\big) \cdot A(U-u)_x + 2\alpha\varepsilon^2 A(U-u)_x \cdot (U-u)_t \Big\} \\
&\quad - \eta''(u) u_x \cdot \Big[F(U) - F(u) - F'(u)(U-u) \Big] \\
&\quad + a_{1t} + a_{2t} + b_{1x} + b_{2x} + 2\alpha\varepsilon (c_{1t} + c_{2t}),
\end{aligned} \tag{5.13}
$$

where

$$
\begin{aligned}
\mathcal{G}(u,U) &= H_R(u,U) \\
&\qquad + \varepsilon^2 \big[\alpha I - \overline{\eta''}\big](U-u)_t \cdot (U-u)_t + \varepsilon^2 \alpha A(U-u)_x \cdot (U-u)_x .
\end{aligned} \tag{5.14}
$$

5.3. The error estimate. Equation (5.13) is the basic decay identity. We see below that, under certain conditions on the entropy η, the quantity $\mathcal{G}(u,U)$ becomes a Lyapunov functional and leads to an error estimate.

PROPOSITION 5.1. *Assume that* (5.1) *is equipped with a strictly convex entropy* η *that satisfies, for some* $\alpha > 0$,

$$
\frac{1}{\alpha} I \le \eta''(u) \le \alpha I, \quad u \in \mathbb{R}^n, \tag{5.15}
$$

and the positive definite, symmetric matrix A *can be selected so that for some* $\nu > 0$ *we have*

$$
\frac{1}{2}\big((\eta''(u)A)^T + \eta''(u)A\big) - \alpha {F'}^T(u) F'(u) \ge \nu I, \quad u \in \mathbb{R}^n. \tag{5.16}
$$

Let u *be a smooth solution of* (5.1), *let* U_ε *be a smooth solution of* (5.2), *and suppose that both* u, U_ε *decay sufficiently fast at infinity.*

(i) *Then* $\mathcal{G}(u,U)$ *is positive definite and*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \mathcal{G}(u,U_\varepsilon)dx + \frac{1}{c}\varepsilon \int_{\mathbb{R}} |(U_\varepsilon - u)_x|^2 + |(U_\varepsilon - u)_t|^2 dx \\
&\le \int_{\mathbb{R}} \Big\{ |\eta''(u)u_x\big(F(U_\varepsilon) - F(u) - F'(u)(U_\varepsilon - u)\big)| \\
&\qquad\qquad + |a_{1t} + a_{2t} + b_{1x} + b_{2x} + 2\alpha\varepsilon(c_{1t} + c_{2t})| \Big\} dx
\end{aligned} \tag{5.17}
$$

for some constant c *independent of* ε.

(ii) *If in addition for some* $M > 0$

$$
|F''(u)| \le M, \quad |\eta'''(u)| \le M, \quad u \in \mathbb{R}^n, \tag{5.18}
$$

then

$$
\begin{aligned}
&\|(U_\varepsilon - u)(t)\|_{L^2} + \varepsilon\|(\partial_x U_\varepsilon - \partial_x u)(t)\|_{L^2} + \varepsilon\|(\partial_t U_\varepsilon - \partial_t u)(t)\|_{L^2} \\
&\quad \le C(t,u)(\|(U_\varepsilon - u)(0)\|_{L^2} + \varepsilon\|(\partial_x U_\varepsilon - \partial_x u)(0)\|_{L^2} + \varepsilon\|(\partial_t U_\varepsilon - \partial_t u)(0)\|_{L^2} + \varepsilon),
\end{aligned} \tag{5.19}
$$

where $C(t,u)$ is a constant depending on t and norms of the smooth solution u.

Proof. Integrating (5.13) over $\mathbb{R}$ and using the hypotheses (5.15) and (5.16), we obtain (5.17). By (5.15),

$$\alpha I - \overline{\eta''} = \alpha I - \int_0^1 \int_0^s \eta''(U + \varepsilon\,\tau\partial_t\,(U-u))d\tau ds \geq \frac{1}{2}\alpha I.$$

Moreover, the function $H_R(u,U)$ defined in (5.6) is strictly convex and thus $\mathcal{G}(u,U)$ in (5.14) is positive definite.

Under (5.15), (5.18) and for

$$\varphi(t) = \int_{\mathbb{R}} |U-u|^2 + \varepsilon^2|U_t - u_t|^2 + \varepsilon^2|U_x - u_x|^2 dx,$$

we have

$$\frac{1}{C}\varphi(t) \leq \int_{\mathbb{R}} \mathcal{G}(u,U)dx \leq C\varphi(t)\,.$$

The error terms in (5.10) are estimated by

$$\|a_{1t}\|_{L^1} \leq \varepsilon C\|u_t\|_{L^\infty}\|U-u\|_{L^2}\|U_t-u_t\|_{L^2}, \qquad \|a_{2t}\|_{L^1} \leq \varepsilon C\|u_{tt}\|_{L^2}\|U-u\|_{L^2},$$
$$\|b_{1x}\|_{L^1} \leq \varepsilon C\|u_x\|_{L^\infty}\|U-u\|_{L^2}\|U_x-u_x\|_{L^2}, \qquad \|b_{2x}\|_{L^1} \leq \varepsilon C\|u_{xx}\|_{L^2}\|U-u\|_{L^2},$$

while the ones in (5.12) are estimated by

$$\|\varepsilon c_{1t}\|_{L^1} \leq \varepsilon^2 C\big(\|u_{tt}\|_{L^2} + \|u_{xx}\|_{L^2}\big)\|U_t-u_t\|_{L^2},$$
$$\|\varepsilon c_{2t}\|_{L^1} \leq \varepsilon C\|u_x\|_{L^\infty}\|U-u\|_{L^2}\|U_t-u_t\|_{L^2},$$

where C is a generic constant depending on α, M, and norms of u.

From (5.17) we obtain

(5.20)
$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \mathcal{G}(u,U_\varepsilon)dx + \frac{1}{C}\varepsilon\left(\|U_t-u_t\|_{L^2}^2 + \|U_x-u_x\|_{L^2}^2\right)\\
&\leq C(\|U-u\|_{L^2}^2 + \varepsilon\|U-u\|_{L^2}(1 + \|U_t-u_t\|_{L^2} + \|U_x-u_x\|_{L^2}) + \varepsilon^2\|U_t-u_t\|_{L^2})\\
&\leq C(\|U-u\|_{L^2}^2 + \varepsilon^2\|U_t-u_t\|_{L^2}^2 + \varepsilon^2\|U_x-u_x\|_{L^2}^2 + \varepsilon^2).
\end{aligned}$$

This in turn gives

$$\varphi(t) \leq \varphi(0) + \varepsilon^2 Ct + C\int_0^t \varphi(s)ds$$

and we conclude from Gronwall's inequality that

$$\varphi(t) \leq C(t,u)\big(\varphi(0) + \varepsilon^2\big). \tag{5.21}$$

Then (5.19) follows. ∎

Remark 5.1. As an example where Proposition 5.1 applies, consider the equations of elastodynamics (4.1). This system admits the entropy pair (4.2). One checks that if

$$0 < s \leq \sigma'(u) \leq S, \quad |\sigma''(u)| \leq M,$$

for some constants s, S, and $M > 0$, then (5.15), (5.16), and (5.18) are fulfilled and we obtain the relevant stability estimate.

Remark 5.2. Proposition 5.1 can be extended for multidimensional hyperbolic systems. In this case, condition (5.16) should be replaced by the analogue of (2.44).

6. Implementation issues. We include here a short discussion on the implementation of the schemes and we present indicative numerical examples that relate to our results.

Adaptivity and mesh reconstruction. The basic principles of our mesh reconstruction policy are

(a) locate the regions of space where increased accuracy is demanded, through a positive functional g;

(b) find a partition of space with predefined constant cardinality and density that follows the estimator function g; and

(c) reconstruct the solution on the finite element space which corresponds to that partition and advance to the next time step by applying the finite element scheme.

These steps are studied, introducing appropriate estimator functions for finite element methods of systems of hyperbolic conservation laws. Among others, estimator functions g are proposed which are based on a posteriori estimates or on the curvature of the approximate solution [4, 2, 3]. This approach yields a dynamic mesh construction which is combined with finite element schemes in what follows, but the mesh selection according to the basic properties of the solution is independent of the particular method used.

Mesh conditions. The mesh conditions needed in the stability analysis in section 3 are somewhat restrictive regarding the flexibility in the selection of the mesh, especially for small values of ε. The main reason is that the time step κ should be chosen very small if ε is very small. (The restrictions on the spatial mesh discussed in Remark 3.1 are not present in the numerical experiments.) In fact, the computational examples show that certain mesh conditions that relate the mesh size and ε are indeed needed and thus for fixed number of spatial mesh points and fixed κ we cannot take ε close to zero; see the following examples and [2, 3].

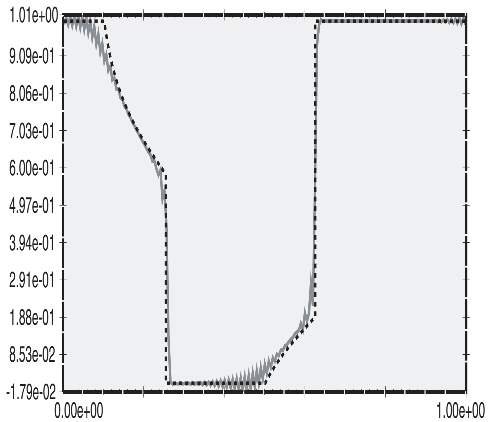
An alternative that completely bypasses this problem is provided by a modification of the finite element relaxation schemes developed in [2, 3]. The alternative is a class of finite element schemes based on the finite element discretization of a modified model with *switched relaxation.* These are schemes in which the application of a Runge–Kutta scheme uses the relaxation finite element model (1.5) for the calculation of the intermediate stages and of u_h^{n+1} and then $v_{h,i}^{n+1}$ is calculated as $v_{h,i}^{n+1} = \Pi F_i(u_h^{n+1})$. This enforces the projection to the equilibrium manifold $v = F(u)$ in each time step. The resulting schemes (switched relaxation finite element schemes) show remarkable stability even for extremely small values of ε. This is illustrated in the examples presented below.

CFL conditions. A common problem in explicit schemes with mesh refinement is to require strong CFL conditions, reflecting the relation of the time step κ to the minimum spatial mesh size $\underline{h}$. This problem appears in the computational examples of [4, 2, 3] but it is not very essential. A computationally more attractive idea would be to use time steps variable with x, or space-time elements, but this will remain for a future work.

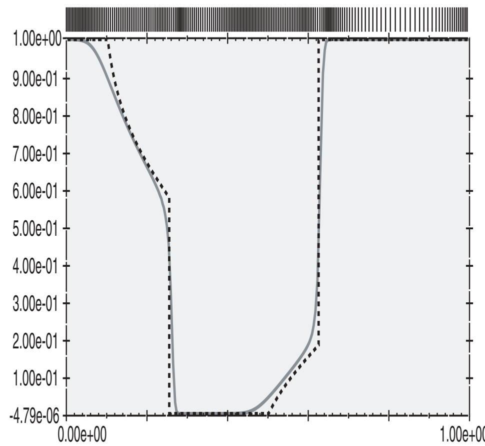
Two-phase flow scalar problem. As a scalar example we chose the Buckley–Leverett equation [30] as a model of a two-phase flow in a porous medium. Here the flux F is not convex and is given by

$$F(u) = \frac{u^2}{u^2 + 0.5(1-u)^2}. \tag{6.1}$$

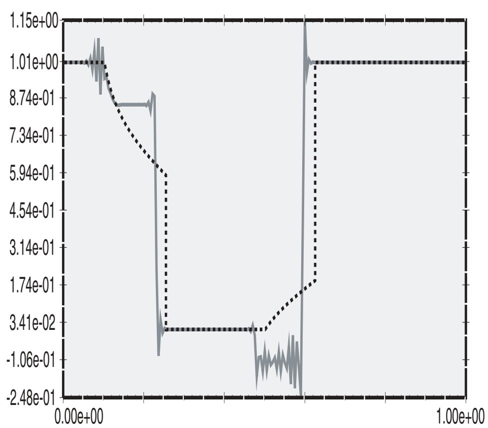
We compute the (periodic) Riemann problem in $[0, 1]$ with $u_0 = 1$ on $[0, 0.1] \cup [0.5, 1]$



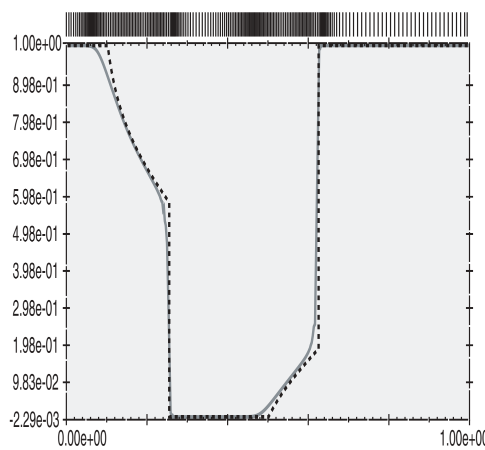
(a) $\varepsilon = 5e{-}4$ uniform mesh



(b) $\varepsilon = 5e{-}4$ refined mesh



(c) $\varepsilon = 5e{-}6$ uniform mesh



(d) $\varepsilon = 5e{-}6$ refined mesh

FIG. 1. *Buckley–Leverett two-phase flow problem:* 200 *nodes on* $[0,1]$. *The effect of the relationship of* h *and* ε *and of the stabilization by mesh refinement. Dotted line: exact solution; gray line: approximation. The distribution of the nodes in the refined mesh is displayed at top in* (b) *and* (d).

and $u_0 = 0$ on $(0.1, 0.5)$. In Figure 1 we display the results of application of our schemes in this problem for 200 nodes in $[0,1]$ with and without mesh refinement. For $\varepsilon = 5e - 4$ the uniform mesh approximation has oscillations, while the corresponding approximation with mesh refinement provides an acceptable solution free of oscillations. Next for $\varepsilon = 5e - 6$ the uniform mesh finite element solution seems to approximate a nonclassical weak solution. Thus the restrictions in our stability results on the relationship of h and ε are necessary. In this case the corresponding finite element approximation with mesh refinement not only eliminates the oscillations but resumes into the approximation of the entropy solution.

It is interesting to note that the method with uniform mesh, although oscillatory, seems to converge (weakly) as $h \to 0$. Moreover, this is also true in the example above where a nonclassical shock for (6.1) is captured. This is an indication that relaxation finite element schemes may conceivably be used to compute nonclassical shocks; compare to [29]. This interesting issue will be examined in a forthcoming work.

For 200 points we cannot take ε smaller unless we use the modified method based on the switched relaxation parameter. In Figure 2 we display the switched relaxation finite element schemes mentioned above. (Here the parameter $\varepsilon = \varepsilon(t)$ is a function of time that vanishes only on discrete time steps and elsewhere has a constant value ε.) Now we can have acceptable approximations for extremely small values of ε. This is a further indication of the strong regularization inherited by the adaptive mesh refinement.

System of elastodynamics. The one-dimensional system of elastodynamics is a particular case where all the results of this paper apply. We consider

$$\begin{aligned} u_{1,t} - u_{2,x} &= 0, \\ u_{2,t} - \sigma(u_1)_x &= 0 \end{aligned}$$

with $\sigma(v) = v + v^3$. We compute the relaxation finite element approximations with Riemann data $u_1(0) = 2$ on $[0, 1/4] \cup [3/4, 1]$ and $u_1(0) = 1$ on $[1/4, 3/4]$ and $u_2(0) = 2$ on $[0, 1]$ extended periodically. Figure 3 displays the approximations for 200 nodes in $[0, 1]$ with mesh refinement for $\varepsilon = 5e-5$. As before we use the modified method with switched relaxation parameter to compute the approximations still with 200 nodes but taking much smaller ε; Figure 4 displays the corresponding results. Figure 5 shows the improvement of the approximations if we use 400 points. In Figure 6 we see the dramatic difference of the approximations with uniform mesh and adaptive mesh refinement still with 400 nodes in $[0, 1]$. For further numerical results and detailed discussion on the adaptive mesh refinement strategies and on implementation issues for the schemes, see [4, 2, 3].

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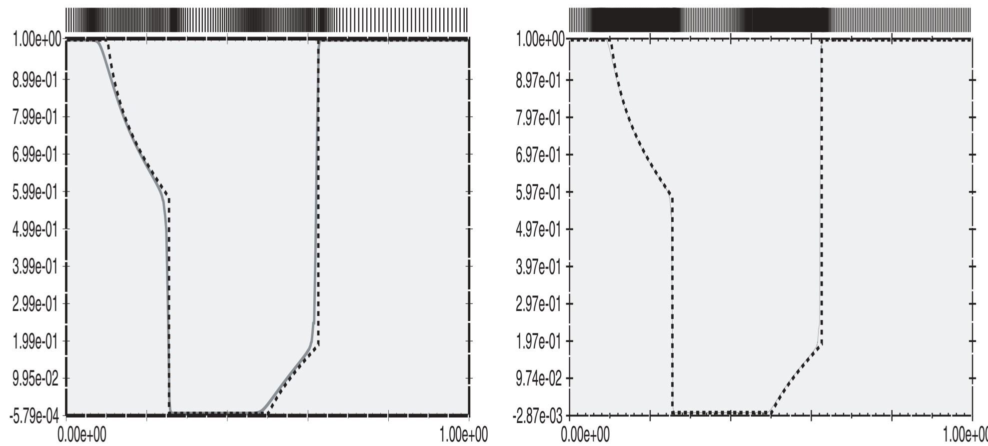
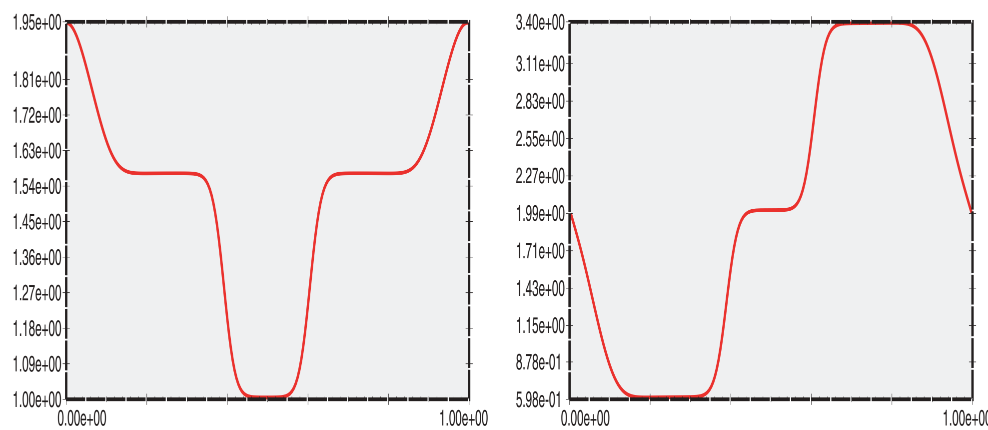


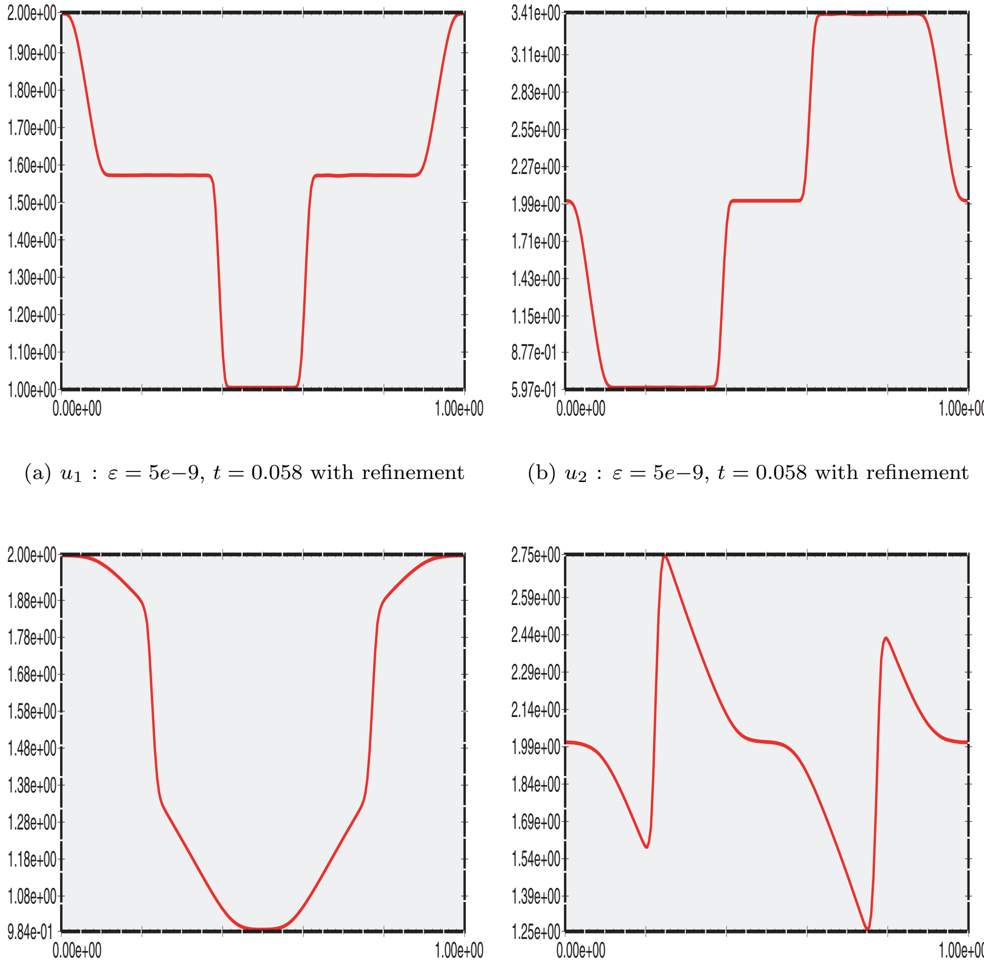
(a) $\varepsilon = 5e{-}7$, $t = 0.112$, 200 nodes (b) $\varepsilon = 5e{-}9$, $t = 0.112$, 400 nodes

Fig. 2. *Buckley–Leverett two-phase flow problem: switched relaxation finite elements with stabilization by mesh refinement. Dotted line: exact solution; gray line: approximation. The distribution of the nodes in the refined mesh is displayed at top.*



(a) u_1 : $\varepsilon = 5e{-}5$, $t = 0.058$, (b) u_2 : $\varepsilon = 5e{-}5$, $t = 0.058$

Fig. 3. *System of elastodynamics:* $q = 1$, 200 *nodes in* $[0, 1]$ *with adaptive mesh refinement.*



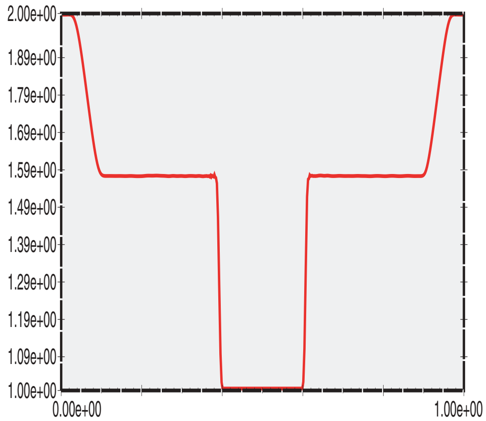
(a) u_1 : $\varepsilon = 5e{-}9$, $t = 0.058$ with refinement

(b) u_2 : $\varepsilon = 5e{-}9$, $t = 0.058$ with refinement

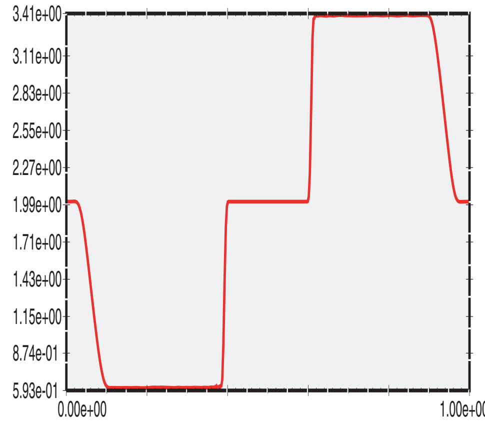
(c) u_1 : $\varepsilon = 5e{-}9$, $t = 0.35$ with refinement

(d) u_2 : $\varepsilon = 5e{-}9$, $t = 0.35$ with refinement

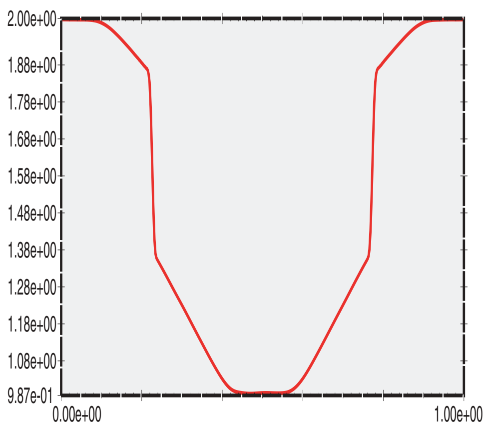
FIG. 4. *System of elastodynamics:* $q = 1$, 200 *nodes in* $[0, 1]$ *with adaptive mesh refinement.*



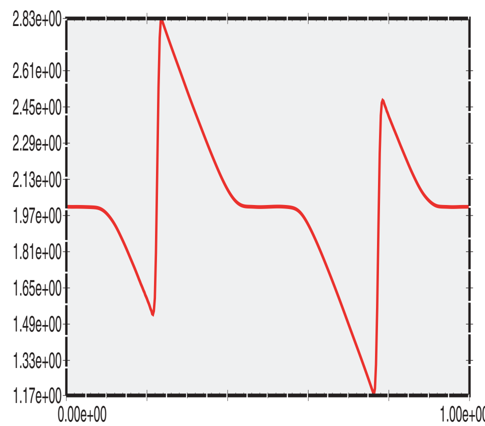
(a) $u_1 : \varepsilon = 5e{-}8$, $t = 0.058$



(b) $u_2 : \varepsilon = 5e{-}8$, $t = 0.058$

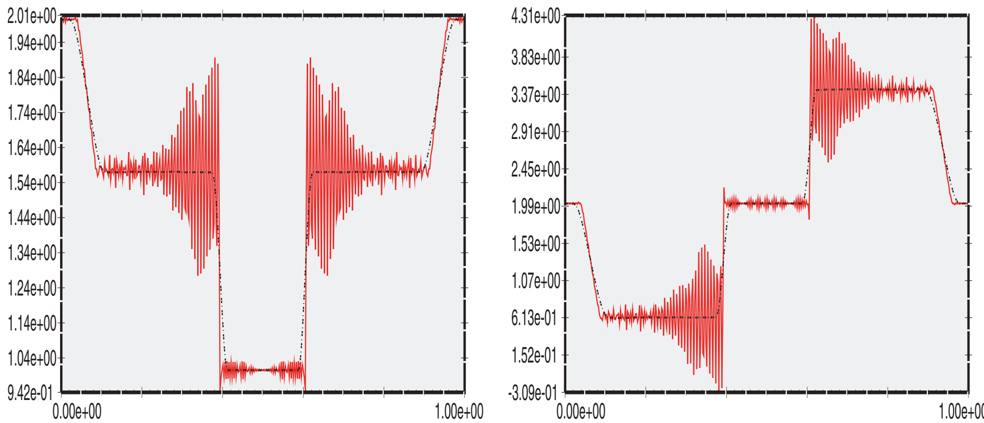


(c) $u_1 : \varepsilon = 5e{-}8$, $t = 0.35$



(d) $u_2 : \varepsilon = 5e{-}8$, $t = 0.35$

FIG. 5. *System of elastodynamics:* $q = 1$, 400 *nodes in* $[0, 1]$ *with adaptive mesh refinement.*



(a) $u_1 : \varepsilon = 5e{-}9$, $t = 0.058$ (b) $u_2 : \varepsilon = 5e{-}9$, $t = 0.058$

FIG. 6. *System of elastodynamics:* $q = 1$, 400 *nodes in* $[0, 1]$ *with uniform mesh (solid lines) and adaptive mesh refinement (dotted lines).*

REFERENCES

[1] D. Aregba-Driollet and R. Natalini, *Discrete kinetic schemes for multidimensional systems of conservation laws*, SIAM J. Numer. Anal., 37 (2000), pp. 1973–2004.

[2] Ch. Arvanitis, *Finite Elements for Hyperbolic Systems of Conservation Laws: New Methods and Computational Techniques*, Ph.D. thesis, University of Crete, Crete, Greece, 2002.

[3] Ch. Arvanitis, *Mesh Refinement Strategies and Finite Element Schemes for Hyperbolic Systems of Conservation Laws*, preprint, University of Crete, Crete, Greece, 2004.

[4] Ch. Arvanitis, Th. Katsaounis, and Ch. Makridakis, *Adaptive finite element relaxation schemes for hyperbolic conservation laws*, Math. Model. Numer. Anal., 35 (2001), pp. 17–33.

[5] I. Babuška and J. E. Osborn, *Analysis of finite element methods for second order boundary value problems using mesh dependent norms*, Numer. Math., 34 (1980), pp. 41–62.

[6] I. Babuška, J. E. Osborn, and J. Pitkäranta, *Analysis of mixed methods using mesh dependent norms*, Math. Comp., 35 (1980), pp. 1039–1062.

[7] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.

[8] G.-Q. Chen, C. D. Levermore, and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math., 47 (1994), pp. 789–830.

[9] B. Cockburn and P.-A. Gremaud, *Error estimates for finite element methods for scalar conservation laws*, SIAM J. Numer. Anal., 33 (1996), pp. 522–554.

[10] B. Cockburn, S. Hou, and C.-W. Shu, *The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws.* IV. *The multidimensional case*, Math. Comp., 54 (1990), pp. 545–581.

[11] B. Cockburn, C. Johnson, C.-W. Shu, and E. Tadmor, *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, A. Quarteroni, ed., Lecture Notes in Math. 1697, Springer-Verlag, New York, 1998.

[12] B. Cockburn and C.-W. Shu, *TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws.* II. *General framework*, Math. Comp., 52 (1989), pp. 411–435.

[13] M. Crouzeix and V. Thomée, *On the stability in L_p and W_p^1 of the L_2 projection into finite element function spaces*, Math. Comp., 48 (1987), pp. 521–532.

[14] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Grundlehren Math. Wiss. 325, Springer-Verlag, Berlin, 2000.

[15] R. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Ration. Mech. Anal., 60 (1983), pp. 75–100.

[16] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems* II: *Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$*, SIAM J. Numer. Anal., 32 (1995), pp. 706–740.
[17] E. Godlewski and P.-A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Appl. Math. Ser. 118, Springer-Verlag, Berlin, 1996.
[18] L. Gosse and A. Tzavaras, *Convergence of relaxation schemes to the equations of elastodynamics*, Math. Comp., 70 (2001), pp. 555–577.
[19] J. Jaffré, C. Johnson, and A. Szepessy, *Convergence of the discontinuous Galerkin finite element method for hyperbolic conservation laws*, Math. Models Methods Appl. Sci., 5 (1995), pp. 367–386.
[20] S. Jin and Z. Xin, *The relaxing schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math., 48 (1995), pp. 235–277.
[21] C. Johnson and A. Szepessy, *On the convergence of a finite element method for a nonlinear hyperbolic conservation law*, Math. Comp., 49 (1987), pp. 427–444.
[22] C. Johnson and A. Szepessy, *Adaptive finite element methods for conservation laws. Part* I: *The general approach*, Comm. Pure Appl. Math., 48 (1995), pp. 199–234.
[23] C. Johnson, A. Szepessy, and P. Hansbo, *On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws*, Math. Comp., 54 (1990), pp. 107–129.
[24] T. Katsaounis and Ch. Makridakis, *Finite volume relaxation schemes for multidimensional conservation laws*, Math. Comp., 70 (2001), pp. 533–553.
[25] M.A. Katsoulakis, G.T. Kossioris, and Ch. Makridakis, *Convergence and error estimates of relaxation schemes for multidimensional conservation laws*, Comm. Partial Differential Equations, 24 (1999), pp. 395–424.
[26] D. Kröner, *Numerical Schemes for Conservation Laws*, John Wiley, New York, 1997.
[27] A. Kurganov and E. Tadmor, *New high resolution central schemes for nonlinear conservation laws and convection-diffusion equations*, J. Comput. Phys., 160 (2000), pp. 241–282.
[28] Ph.G. LeFloch, *Hyperbolic systems of conservation laws. The theory of classical and nonclassical shock waves*, Lectures in Mathematics ETH Zurich, Birkhäuser Verlag, Basel, Switzerland, 2002.
[29] Ph.G. LeFloch and C. Rohde, *High-order schemes, entropy inequalities, and nonclassical shocks*, SIAM J. Numer. Anal., 37 (2000), pp. 2023–2060.
[30] R.J. LeVeque, *Numerical Methods for Conservation Laws*, Lectures Math. ETH Zurich, Birkhäuser, Zurich, 1992.
[31] P. Lin, *Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics*, Trans. Amer. Math. Soc., 329 (1992), pp. 377–413.
[32] F. Murat, *L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$*, J. Math. Pures Appl., 60 (1981), pp. 309–322.
[33] H. Nessyahu and E. Tadmor, *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comput. Phys., 87 (1990), pp. 408–463.
[34] B. Perthame and A.E. Tzavaras, *Kinetic formulation for systems of two conservation laws and elastodynamics*, Arch. Ration. Mech. Anal., 155 (2000), pp. 1–48.
[35] M.E. Schonbek, *Convergence of solutions to nonlinear dispersive equations*, Comm. Partial Differential Equations, 7 (1982), pp. 959–1000.
[36] D. Serre, *Relaxation semi linéaire et cinétique des systèmes de lois de conservation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), pp. 169–192.
[37] D. Serre and J. Shearer, *Convergence with Physical Viscosity for Nonlinear Elasticity*, manuscript, 1993.
[38] J.W. Shearer, *Global existence and compactness in L^p for the quasi-linear wave equation*, Comm. Partial Differential Equations, 19 (1994), pp. 1829–1877.
[39] C.-W. Shu, *Total-variation-diminishing time discretizations*, SIAM J. Sci. Statist. Comput., 9 (1988), pp. 1073–1084.
[40] C.-W. Shu and S. Osher, *Efficient implementation of essentially nonoscillatory shock-capturing schemes*, J. Comput. Phys., 77 (1988), pp. 439–471.
[41] A. Szepessy, *Convergence of a shock-capturing streamline diffusion finite element method for a scalar conservation law in two space dimensions*, Math. Comp., 53 (1989), pp. 527–545.
[42] L. Tartar, *Compensated compactness and applications to partial differential equations*, in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV, R. J. Knops, ed., Res. Notes in Math. 39, Pitman, Boston, 1979, pp. 136–212.
[43] A. Tzavaras, *Materials with internal variables and relaxation to conservation laws*, Arch. Ration. Mech. Anal., 146 (1999), pp. 129–155.