KINETIC FORMULATION FOR SYSTEMS OF TWO CONSERVATION LAWS AND ELASTODYNAMICS

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Abstract. For scalar conservation laws, the kinetic formulation is a way to generate all the entropies by a simple kernel. We show how this concept replaces and simplifies greatly the concept of Young measures avoiding the difficulties encountered when working in L^p. The general construction of the two kinetic functions that generate the entropies of 2×2 strictly hyperbolic systems is also developed here. We show that it amounts to build a 'universal' entropy i.e. that can be truncated by a 'kinetic value' along Riemann invariants. For elastodynamics, this construction can be completed and specialized using the additional Galilean invariance. This allows a full characterization of convex entropies. It yields a kinetic formulation consisting of two semi-kinetic equations which, as usual, are equivalent to the infinite family of all the entropy inequalities.

Contents

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1. Introduction

The so-called *kinetic formulation* of nonlinear hyperbolic systems of conservation laws is a method which reduces them to a linear equation, with an additional *kinetic* variable, on a nonlinear quantity related to the conserved unknowns. It represents all the entropy inequalities in a single equation depending on an additional variable. It was introduced on two examples by P.L. Lions, B. Perthame and E. Tadmor : scalar conservation laws (see [LPT_1]) and isentropic gas dynamics with a γ law (see [LPT_2]). It turns out to be a powerful tool to derive mathematical properties -and also numerical schemes, although this aspect will not be treated here. A new class of $L^p_x(L^1_t)$ estimates has been proved in these works. Compensated compactness arguments for existence of solutions (see [LPS]), appear simpler than in the original setting of R. DiPerna [Dp_1], [Dp_2] and extend it to the class of pressure laws left open for instance in Ding, Chen and Luo [DCL]. For the scalar case, it turns out to be a powerful method to prove regularizing effects in Sobolev spaces by averaging lemma arguments. The method was subsequently used by several authors, who gave further examples of kinetic formulations : for an $n\times n$ system of chromatography (see James, Peng and Perthame [JPP]), general pressure laws in isentropic gas dynamics (see Chen and LeFloch [CL]). Initial boundary value problems have been treated in Nouri, Omrane and Vila [NOV], and applications to time continuity are presented in Vasseur [Va].

The purpose of the present article is twofold. Firstly, we explain how the concept of kinetic function replaces and simplifies greatly the concept of Young measures, by avoiding some of the topological difficulties encountered when working with Young measures in L^p. We also give a simple extension of the results of Tartar [Ta] and Schonbek [S] on compactness of approximate solutions for scalar conservation laws, to the case that the approximations are relatively weakly compact in L^1_{loc}. The idea behind is very simple. While Young measures are objects in the space of L^p functions with values in Radon measures, the kinetic function simply belongs to L^∞, and both allow to represent nonlinear functions by integrals, see (1.4) below for the more complex case of a system. Secondly, we give a systematic construction of the kinetic functions that generate the entropies of 2×2 strictly hyperbolic systems. We show that they are built based on some kind of 'universal' entropy with the special property that it can be truncated by a 'kinetic value' along one of the Riemann invariants.

Then we specialize our construction to the case of the system of elastodynamics

$$
\begin{aligned}
u_t - v_x &= 0 \ , \\
v_t - \sigma(u)_x &= 0 \ ,
\end{aligned}
\tag{1.1}
$$

under the hypotheses that the stress-strain function $\sigma(u)$ be twice continuously differentiable, $\sigma'(u) > 0$, and $u\sigma''(u) > 0$ for $u \neq 0$. We turn attention to characterizing the convex entropies, solutions of the linear wave equation

$$\eta_{uu} = a^2(u)\eta_{vv}\,, \qquad \text{with } \ a(u) = \big(\sigma'(u)\big)^{1/2}. \tag{1.2}$$

Our analysis is completed using two additional ingredients: The Galilean invariance of (1.1) and a characterization of *convex* solutions for (1.2) observed in Dafermos [Da_1] and further developed here. This allows a full characterization of convex entropies which yields a kinetic formulation. It consists of two semi-kinetic equations which, as usual, are equivalent to the infinite family of all the entropy inequalities. More precisely, there are two kinetic functions (which replace the Maxwellian distribution in the classical Boltzmann theory), $\Theta_o(u,v,\xi)$ and $\bar\Theta_o(u,v,\xi)$, such that entropy solutions to (1.1) satisfy for some positive, bounded measures m, $\bar m$ on $\mathbb{R}_x \times \mathbb{R}_t^+ \times \mathbb{R}_\xi$, the equations

$$\begin{aligned}
&\partial_t\Theta_o(u,v,\xi) + \partial_x\Big[\big(-a(u)\kappa_o + a(u)(1-\kappa_o)\big)\Theta_o(u,v,\xi)\Big] = \partial_\xi m(x,t,\xi)\ ,\\
&\partial_t\bar\Theta_o(u,v,\xi) + \partial_x\Big[\big(-a(u)\bar\kappa_o + a(u)(1-\bar\kappa_o)\big)\bar\Theta_o(u,v,\xi)\Big] = \partial_\xi \bar m(x,t,\xi)\ ,
\end{aligned} \tag{1.3}$$

in the sense of distributions in $\mathcal{D}'_{x,t,\xi}$. Here $a(u)$ is the sound speed, and $\kappa_o(u,v,\xi)$, $\bar\kappa_o(u,v,\xi)$ are given functions satisfying $0 \le \kappa_o\,,\bar\kappa_o \le 1$. They have the effect that the speeds of propagation in (1.3) cover the full range $[-a(u),\ a(u)]$ between the characteristic speeds of the elastodynamics system. Also, a remarkable property of the system is that $\Theta_o(u,v,\xi)$ and $\bar\Theta_o(u,v,\xi)$ are bounded and discontinuous along Riemann invariants, with bounded support also related to the speeds of propagation. Convex entropies of the system (1.1) are just obtained by the formula

$$\eta(u,v) = \int_{\mathbb{R}} \Theta_o(u,v,\xi)\partial_\xi p(\xi)d\xi + \int_{\mathbb{R}} \bar\Theta_o(u,v,\xi)\partial_\xi \bar p(\xi)d\xi\ , \tag{1.4}$$

with p, $\bar p$ convex. We hope that this construction, not only is useful to give direct and simple proofs of many properties of the system, as we do it below, but will give a first step towards the discovery of regularizing effects.

The article is organized as follows. In Section 2, we recall the scalar case and outline the fundamental property of the kinetic function to replace Young measures and illustrate it by proving a strong compactness result. In Section 3, we give a general construction of discontinous kinetic entropies for 2×2 hyperbolic systems and prove that they indeed generate all the entropies of the system. Sections 4 and 5 are devoted to the special case of elastodynamics. We characterize the convex entropies completely thus proving (1.4). In the last section we introduce the kinetic formulation (1.3) and some consequences.

2. The Scalar Conservation Law

The first section concerns the scalar conservation law in one space dimension

$$u_t + f(u)_x = 0\,. \tag{2.1}$$

Our objective is to introduce some notations which will be of current use throughout the paper, to outline the kinetic formulation for scalar conservation laws of Lions, Perthame and Tadmor [LPT$_1$] and to present a new variant of the proof of the cancellations of oscillations for approximate solutions of (2.1), of Tartar [Ta] and Schonbek [S]. In particular, we outline that the kinetic function contains a microscopic information which avoids using the Young measures (in the L^p setting of Ball [Ba]) and simplifies the analysis of weak limits.

2.a A class of indicator functions.

In preparation, we introduce a class of indicator functions familiar from the kinetic formulation of scalar conservation laws. Let $\mathbb{1}_u(\xi) = \mathbb{1}(u,\xi)$ be defined by

$$\mathbb{1}_u(\xi) = \mathbb{1}(u,\xi) := \begin{cases} \mathbb{1}_{0<\xi<u} & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -\mathbb{1}_{u<\xi<0} & \text{if } u < 0. \end{cases} \tag{2.2}$$

The function $\mathbb{1}_u(\xi)$ satisfies in $\mathcal{D}'$

$$\begin{aligned} &\partial_u \mathbb{1}_u(\xi) = \delta(u-\xi)\,, \\ &\partial_\xi \mathbb{1}_u(\xi) = -\delta(u-\xi) + \delta(\xi)\,. \end{aligned} \tag{2.3}$$

It also serves to define an extended class of indicator functions $\mathbb{1}_u^k(\xi)$ by

$$\mathbb{1}_u^k(\xi) = \mathbb{1}_{u-k}(\xi - k) = \begin{cases} \mathbb{1}_{k<\xi<u} & \text{if } k < u \\ 0 & \text{if } k = u \\ -\mathbb{1}_{u<\xi<k} & \text{if } u < k \end{cases} \tag{2.4}$$

The interest in the latter lies in that they encompass in one framework (2.2) together with the limiting cases $k = \pm\infty$, when $\mathbb{1}_u^k(\xi)$ take the form

$$\mathbb{1}_u^{-\infty}(\xi) = \mathbb{1}_{\xi<u}\,, \quad \mathbb{1}_u^{\infty}(\xi) = -\mathbb{1}_{u<\xi}. \tag{2.5}$$

It is easy to show the following properties: If the signum function is defined by

$$\text{sign } x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases}$$

then

$$\mathbb{1}_u^k(\xi) = \text{sign } (u-k)\ \mathbb{1}_{\{\xi\in(u,k)\}}. \tag{2.6}$$

For any k finite, it is

$$\begin{aligned} &\partial_u \mathbb{1}_u^k(\xi) = \delta(u-\xi)\,, \\ &\partial_\xi \mathbb{1}_u^k(\xi) = -\delta(u-\xi) + \delta(\xi - k)\,. \end{aligned}$$

Finally, we have the formula

$$h(u) - h(k) = \begin{cases} \int_k^u h'(\xi)\, d\xi & \text{for } k < u \\ -\int_u^k h'(\xi)\, d\xi & \text{for } u < k \end{cases} = \int_{\mathbb{R}} \mathbb{1}_u^k(\xi)\, h'(\xi)\, d\xi, \tag{2.7a}$$

for $h(u)$ a smooth function and k finite. Also, the related formulas for the limiting cases $k = \pm\infty$ hold when h' is integrable in a neighborhood of $-\infty$. Then

$$h(u) - h(-\infty) = \int_{\mathbb{R}} \mathbb{1}_{\xi<u}\, h'(\xi)\, d\xi = \int_{\mathbb{R}} \mathbb{1}_u^{-\infty}(\xi)\, h'(\xi)\, d\xi, \tag{2.7b}$$

while, if h' is integrable in a neighborhood of $+\infty$, then

$$h(u) - h(\infty) = -\int_{\mathbb{R}} \mathbb{1}_{\xi>u}\, h'(\xi)\, d\xi = \int_{\mathbb{R}} \mathbb{1}_u^{\infty}(\xi)\, h'(\xi)\, d\xi. \tag{2.7c}$$

The first use of (2.7) arises in representation formulas for entropy-entropy flux pairs of (2.1). Let $\lambda(u) = f'(u)$ and recall that any pair of smooth functions $(\eta(u), q(u))$ satisfying

$$q'(u) = \lambda(u)\eta'(u) \tag{2.8}$$

is called an entropy-entropy flux pair. From (2.7) we see that such entropy pairs are given by representation formulas of the form

$$\begin{aligned} \eta(u) &= \int_{\mathbb{R}} \mathbb{1}_u(\xi)\, \phi(\xi) d\xi, \\ q(u) &= \int_{\mathbb{R}} \mathbb{1}_u(\xi)\, \lambda(\xi)\, \phi(\xi) d\xi, \end{aligned} \tag{2.9}$$

where $\phi = \eta'$ can be thought of as a test function.

2.b Propagation and cancellation of oscillations for scalar conservation laws.

Our purpose is to analyze the behavior of a family of approximate solutions $\{u_\varepsilon\}$ of (2.1), that are locally weakly compact in L^1. We recall that a family $\{u^\varepsilon\} \subset L^1(K)$, K of finite measure, is weakly compact if $\{u^\varepsilon\}$ is uniformly integrable, *i.e.*

$$\int_{|u^\varepsilon| \geq c} |u^\varepsilon| dx \to 0 \qquad \text{as } c \to \infty, \text{ uniformly in } \varepsilon$$

Equivalently,

$$\text{if, for some } \Psi : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{nondecreasing and satisfying } \frac{\Psi(\tau)}{\tau} \to \infty \text{ as } \tau \to \infty,$$
$$\Psi(|u^\varepsilon|) \text{ is bounded in } L^1(K).$$

We refer to Dunford-Schwartz [DS, pp. 289-295] and Dellacherie-Meyer [DM, pp. 21-28] for properties of the weak topology in L^1_{loc} and the Dunford-Pettis characterisation of weak L^1_{loc} compactness.

After extraction of subsequences, there is a function $u \in L^1_{\text{loc}}$ such that

$$u^\varepsilon \rightharpoonup u, \quad w - L^1_{\text{loc}}. \tag{2.10}$$

and a function $g(t,x,\xi) \in L^\infty$ which satisfies

$$\mathbb{1}(u^\varepsilon, \xi) \rightharpoonup g, \quad L^\infty - w\star, \tag{2.11}$$

and, from the very definition of weak convergence, we deduce

$$\begin{cases} \ \ 0 \le g(t,x,\xi) \le 1, & \text{for } \xi \ge 0, \\ -1 \le g(t,x,\xi) \le 0, & \text{for } \xi \le 0. \end{cases} \tag{2.12}$$

A first step in our analysis is a simple remark which allows to use g in order to replace the Young measures. It is more convenient because we deal with a $L^1_{\text{loc};\,t,x}(L^1_\xi)$ function, this avoids the difficulties encountered in [Ba] for dealing with weak topologies on measures.

Lemma 2.1. *With the assumptions (2.10 , 2.11), the above function g belongs to $L^1_{\text{loc};\,t,x}(L^1_\xi)$, and satisfies a.e. $(t,\ x)$,*

$$\int_{\mathbb{R}} |g(t,x,\xi)|\, d\xi = w\text{-}lim |u^\varepsilon(t,x)| \ \ \in L^1_{loc}, \tag{2.13}$$

$$\int_{\mathbb{R}} g(t,x,\xi)\, d\xi = u(t,x), \tag{2.14}$$

$$\int_{\mathbb{R}} S'(\xi) g(t,x,\xi)\, d\xi = w\text{-}lim\, S\big(u^\varepsilon(t,x)\big), \quad \forall S' \in L^\infty,\ S(0)=0, \tag{2.15}$$

Moreover,

$$w\text{-}lim\, S\big(u^\varepsilon(t,x)\big) = S(u)\,,\ \forall S' \in L^\infty,\ S(0)=0, \quad \textit{if and only if} \quad g = \mathbb{1}(u,\xi)\,. \tag{2.16}$$

Proof. First notice that from (2.11) and Fatou's lemma, for K a compact subset of $\mathbb{R} \times \mathbb{R}^+$,

$$\begin{aligned} \int_K \int_{\mathbb{R}} |g| &\le \int_K \int_{\mathbb{R}} \liminf |\mathbb{1}(u^\varepsilon,\xi)| \\ &\le \liminf \int_K \int_{\mathbb{R}} \text{sign } \xi\, \mathbb{1}(u^\varepsilon,\xi) = \liminf \int_K |u^\varepsilon| \end{aligned}$$

and thus $g \in L^1_{loc;t,x}(L^1_\xi)$.

Then, we only prove the identity (2.14), the other identities being simple variants. We introduce two real numbers, $R > 0$ (large) and $S < 0$ (small), we have

$$u^\varepsilon = \int_S^R \mathbb{1}(u^\varepsilon,\xi)\,d\xi + \int_R^{+\infty} \mathbb{1}(u^\varepsilon,\xi)\,d\xi + \int_{-\infty}^S \mathbb{1}(u^\varepsilon,\xi)\,d\xi. \tag{2.17}$$

But, from the definition of $\mathbb{1}(u^\varepsilon,\xi)$, we deduce, after further extraction,

$$\int_R^{+\infty} \mathbb{1}(u^\varepsilon,\xi)\,d\xi = (u^\varepsilon - R)_+ \rightharpoonup v^R(t,x), \quad w - L^1_{\mathrm{loc}},$$

and from the Dunford-Pettis characterisation of weak compactness in L^1, we deduce that, as $R \to \infty$,

$$v^R(t,x) \rightharpoonup 0, \quad w - L^1_{\mathrm{loc}}. \tag{2.18}$$

In the same way, we find a function v_S which also satisfies (2.18), and passing to the limit in (2.17), we find

$$u(t,x) = v^R(t,x) + v_S(t,x) + \int_S^R g(t,x,\xi)\,d\xi.$$

Passing to the limit in R and S and using that $g \in L^1_{loc;t,x}(L^1_\xi)$ we recover (2.14). The last statement of the Lemma is a direct consequence of (2.15). □

Remark 2.2.

1. Note the relation of g with the Young (probabilty) measure μ associated with u^ε

$$\partial_\xi g(t,x,\xi) = \delta(\xi = 0) - \mu_{(t,x)}(\xi).$$

2. Let S be a convex function, $\alpha \in \mathbb{R}$ and consider the minimization problem

$$\inf \int_{\mathbb{R}} S'(\xi) f(\xi)\,d\xi \quad \text{over } f \in L^1(\mathbb{R}) \text{ such that } \begin{cases} 0 \le f \le 1 & \xi > 0 \\ -1 \le f \le 0 & \xi < 0 \end{cases}$$
$$\text{and } \int_{\mathbb{R}} f = \alpha$$

Brenier [Br] showed that this minimization problem achieves its minimum at $f = \mathbb{1}(\alpha,\xi)$, and that if S is strictly convex the minimizer is unique.

As an implication, for S stricly convex with $S' \in L^\infty$, $S(0) = 0$, we have

$$\text{w-lim } S\big(u^\varepsilon(t,x)\big) = \int_{\mathbb{R}} S'(\xi) g(t,x,\xi)\,d\xi \ge \int_{\mathbb{R}} S'(\xi)\mathbb{1}(u(t,x),\xi) = S(u) \tag{2.19}$$

with equality if and only if $g(t,x,\xi) = \mathbb{1}(u(t,x),\xi)$. Notice that, when formulated in terms of Young measures, (2.19) amounts to Jensen's inequality.

In the sequel, we use the function $g_+(t,x,\xi)$ defined (along a subsequence, if necessary) by

$$\mathbb{1}_{\xi<u^\varepsilon} \rightharpoonup g_+, \quad L^\infty - w\star, \tag{2.20}$$

Note that $0 \le g_+ \le 1$ and that, since $\mathbb{1}_{\xi<u^\varepsilon}$ is strictly decreasing in ξ, the function g is decreasing in ξ as well. By (2.2), the functions g and g_+ are connected by

$$g = g_+ \mathbb{1}_{\xi>0} - (1-g_+)\mathbb{1}_{\xi<0} \quad \text{a.e. } t, x, \xi\,. \tag{2.21}$$

We are now ready to state our main result.

Theorem 2.3. *Assume that the flux f is C^2 and let $\{u^\varepsilon\}$ satisfy (2.10, 2.20), and*

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \quad \text{lies in a compact of } H^{-1}_{loc} \tag{2.22}$$

for any (η, q) with $\eta_u \in C^1_c(\mathbb{R})$. If $I = \{\xi \in \mathbb{R} : 0 < g_+(x,t,\xi) < 1\}$, then for a.e. (x,t) :
(i) either I an interval in which case the speed $\lambda(\xi)$ must remain constant on I;
(ii) or I is empty or a single point in which case $g = \mathbb{1}(u,\xi)$.

Proof. Following the argument of Tartar for compensated compactnes, we consider the classes of entropy-entropy flux pairs of the type (2.7b), (2.7c):

$$\eta_1(u) = \int_{\mathbb{R}} \mathbb{1}_{\xi<u}\phi(\xi)d\xi, \qquad \eta_2(u) = \int_{\mathbb{R}} \mathbb{1}_{u<\theta}\psi(\theta)d\theta,$$
$$q_1(u) = \int_{\mathbb{R}} \mathbb{1}_{\xi<u}\lambda(\xi)\phi(\xi)d\xi, \qquad q_2(u) = \int_{\mathbb{R}} \mathbb{1}_{u<\theta}\lambda(\theta)\psi(\theta)d\theta,$$

where ϕ, $\psi \in C^1_c(\mathbb{R})$ and we recall that $\lambda = f'$. Both pairs are constant near infinity. We denote with brackets the various weak limits, in (t,x) or in (t,x,ξ), and apply the usual compensated compactness identity:

$$< \eta_1 q_2 - \eta_2 q_1 > = < \eta_1 >< q_2 > - < \eta_2 >< q_1 > .$$

Using the representation formulas

$$\eta_1(u^\varepsilon) \rightharpoonup \int_{\mathbb{R}} < \mathbb{1}_{\xi<u^\varepsilon} > \phi(\xi)d\xi\,, \quad \eta_2(u^\varepsilon) \rightharpoonup \int_{\mathbb{R}} < \mathbb{1}_{u^\varepsilon<\theta} > \psi(\theta)d\theta$$

and so on, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \big(\lambda(\theta)-\lambda(\xi)\big)\Big[< \mathbb{1}_{\xi<u^\varepsilon}\mathbb{1}_{u^\varepsilon<\theta} > - < \mathbb{1}_{\xi<u^\varepsilon} >< \mathbb{1}_{u^\varepsilon<\theta} > \Big]\phi(\xi)\psi(\theta)\, d\xi d\theta = 0,$$

from where we conclude

$$(2.23)\qquad (\lambda(\theta)-\lambda(\xi))\Big[< \mathbb{1}_{\xi<u^\varepsilon}\mathbb{1}_{u^\varepsilon<\theta} > - < \mathbb{1}_{\xi<u^\varepsilon} > < \mathbb{1}_{u^\varepsilon<\theta} > \Big] = 0 \quad \text{a.e. } \xi,\theta.$$

Recalling that $g_+(t,x,\xi) =< \mathbb{1}_{\xi<u^\varepsilon} >$, we have for $\theta<\xi$

$$[\lambda(\theta)-\lambda(\xi)]\, g_+(\xi)\big(1-g_+(\theta)\big) = 0\,.$$

The function g_+ is decreasing and, due to the fact that $g\in L^1_\xi$ and (2.21), we have $g_+(-\infty)=1$ and $g_+(\infty)=0$. The set I is either empty, or a single point, or an interval. If I is an interval and $\theta<\xi$ any interior points then $\lambda(\theta)=\lambda(\xi)$ and thus λ stays constant on I. If I is empty or a single point, then $g_+(\xi)=\mathbb{1}_{\xi<a}$ for some $a\in\mathbb{R}$, and we conclude from (2.21), (2.2) and (2.14) that $a=u$ and $g=\mathbb{1}(u,\xi)$. □

3. Singular Entropies for Systems of two Conservation Laws

We consider a system of two conservation laws

$$(3.1)\qquad \begin{aligned} u_t + a(u,v)_x &= 0\\ v_t + b(u,v)_x &= 0\end{aligned}$$

The flux functions are smooth and such that the system is strictly hyperbolic with characteristic speeds $\lambda_1(u,v)<\lambda_2(u,v)$. The right eigenvectors $r_1(u,v)$, $r_2(u,v)$ and left eigenvectors $l_1(u,v)$, $l_2(u,v)$ are linearly independent and normalized so that $r_i(u,v)\cdot l_j(u,v)=\delta_{ij}$.

Let $U=(u,v)$ be the conserved variable and $F(U)=(a(u,v),b(u,v))$ denote the flux-function. A scalar-valued function $\eta(U)$ is called an entropy with corresponding entropy flux $q(U)$ if every smooth solution of the conservation law (3.1) satisfies the additional conservation law

$$(3.2)\qquad \partial_t\eta(U)+\partial_x q(U) = 0\,.$$

Entropy pairs $\eta(U)-q(U)$ are connected through the differential relations

$$(3.3)\qquad \nabla q(U) = \nabla\eta(U)\cdot\nabla F(U)\,,$$

where ∇ denotes the gradient with respect to (u,v). For systems of two conservation laws, (3.3) is a determined system. In this section we construct certain singular entropy pairs that turn out to generate the general solution of (3.3). The process naturally gives rise to two parametrized families of singular entropies which are closely related to a kinetic formulation.

3.a Riemann invariants and entropies.

Let $w = w(u,v)$ and $z = z(u,v)$ be the 1- and 2-Riemann invariants respectively, defined by $\nabla w = l_1$ and $\nabla z = l_2$, or equivalently

$$\nabla w \cdot r_2 = 0\,, \quad \nabla w \cdot r_1 = 1\,,$$
$$\nabla z \cdot r_2 = 1\,, \quad \nabla z \cdot r_1 = 0\,.$$

For systems of two conservation laws, the Riemann invariants are well defined, at least locally, and induce a transformation $T : (u,v) \to (w,z)$ which is one-to-one and invertible in a neighborhood of each point (u_0, v_0). For certain special systems, like the equations of elastodynamics, T can be a globally well defined and invertible map.

Any given field ψ may be expressed in terms of the state vector (u,v) or in terms of the Riemann invariants (w,z), according to the formula

$$\psi(u,v) = \hat{\psi}(w(u,v), z(u,v))\,,$$
$$\frac{\partial \hat{\psi}}{\partial w} = (r_1 \cdot \nabla)\psi\,, \quad \frac{\partial \hat{\psi}}{\partial z} = (r_2 \cdot \nabla)\psi\,.$$

Henceforth, with a slight abuse of notation, we will retain the same symbol ψ for both expressions of the field. These formulas are useful for expressing various properties in terms of the variables (w,z): For instance, genuine nonlinearity for the first characteristic field is expressed as

$$r_1 \cdot \nabla \lambda_1 > 0 \quad \text{or as} \quad \frac{\partial \lambda_1}{\partial w} > 0\,. \tag{3.4a}$$

Genuine nonlinearity of the second characteristic field is expressed as

$$r_2 \cdot \nabla \lambda_2 > 0 \quad \text{or as} \quad \frac{\partial \lambda_2}{\partial z} > 0\,. \tag{3.4b}$$

Finally, in a coordinate system of Riemann invariants, the equation for the entropies (3.3) takes the form

$$\begin{aligned} q_w &= \lambda_1 \eta_w\,, \\ q_z &= \lambda_2 \eta_z\,. \end{aligned} \tag{3.5}$$

By differentiating (3.3), one can show that for a strictly hyperbolic system

$$r_1 \cdot \nabla^2 \eta\, r_2 = r_2 \cdot \nabla^2 \eta\, r_1 = 0\,.$$

Accordingly, strict convexity of the entropy $\eta(u,v)$ amounts to

$$r_1 \cdot \nabla^2 \eta\, r_1 > 0\,, \quad r_2 \cdot \nabla^2 \eta\, r_2 > 0\,.$$

These relations can also be expressed in coordinates of Riemann invariants (*c.f.* Dafermos [Da$_2$]) by the formulas

$$r_1 \cdot \nabla^2 \eta \, r_1 = \partial_w^2 \eta + \big(r_1 \cdot \nabla^2 w \, r_1\big)\partial_w \eta + \big(r_1 \cdot \nabla^2 z \, r_1\big)\partial_z \eta \,, \qquad (3.6)$$
$$r_2 \cdot \nabla^2 \eta \, r_2 = \partial_z^2 \eta + \big(r_2 \cdot \nabla^2 w \, r_2\big)\partial_w \eta + \big(r_2 \cdot \nabla^2 z \, r_2\big)\partial_z \eta \,.$$

Entropy pairs (η, q) are constructed by solving the second order hyperbolic equation

$$\eta_{wz} = \frac{\lambda_{2w}}{\lambda_1 - \lambda_2}\, \eta_z - \frac{\lambda_{1z}}{\lambda_1 - \lambda_2}\, \eta_w \,. \qquad (3.7)$$

Given a solution η of (3.7) then (3.5) becomes exact and q is obtained by integration.

It is instructive to review a slight variation of the above construction. Let us introduce the functions

$$f(w,z) = e^{\int^w \frac{\lambda_{2w}}{\lambda_1 - \lambda_2} ds} \,, \qquad g(w,z) = e^{-\int^z \frac{\lambda_{1z}}{\lambda_1 - \lambda_2} ds} \,, \qquad (3.8a)$$

solutions of the equations

$$f_w = \frac{\lambda_{2w}}{\lambda_1 - \lambda_2}\, f \,, \qquad g_z = -\frac{\lambda_{1z}}{\lambda_1 - \lambda_2}\, g \,. \qquad (3.8b)$$

Note that f and g are defined within a multiplicative factor (reflecting in the lower limits of integration in (3.8a)), and their role will be clarified later when constructing singular entropies. Then (3.7) takes the form

$$\eta_{wz} = \frac{g_z}{g}\eta_w + \frac{f_w}{f}\eta_z \,. \qquad (3.9)$$

It is possible to construct entropy-entropy flux by the following process: With $\varphi = \eta_w$ and $\psi = \eta_z$ we first solve the system

$$\varphi_z = \frac{g_z}{g}\, \varphi + \frac{f_w}{f}\, \psi \,, \qquad (3.10)$$
$$\psi_w = \frac{g_z}{g}\, \varphi + \frac{f_w}{f}\, \psi \,.$$

Given a solution (φ, ψ) of (3.10), the associated entropy pair (η, q) is obtained by integrating the systems $\eta_w = \varphi$, $\eta_z = \psi$ and $q_w = \lambda_1 \varphi$, $q_z = \lambda_2 \psi$. Equations (3.10) guarantee that both these systems are exact and the quadrature problems admit solutions.

3.b Singular entropies.

In this section we construct distributional solutions to the entropy equations (3.5). These solutions are expressed using the indicator functions in Section 2.a, and will provide a representation formula for solutions of the Goursat problem. The analysis is presented in terms of the functions $\mathbb{1}_w(\xi)$ and $\mathbb{1}_z(\zeta)$. Similar statements hold if the functions $\mathbb{1}_w^k(\xi)$ and $\mathbb{1}_z^k(\zeta)$ are used instead, for any value of the parameter k including the interesting limit cases $k = \pm\infty$. In the limit cases, $\mathbb{1}_w^{\infty}(\xi) = \mathbb{1}_{\xi < w}$, $\mathbb{1}_w^{-\infty}(\xi) = -\mathbb{1}_{w<\xi}$, $\mathbb{1}_z^{\infty}(\zeta) = \mathbb{1}_{\zeta < z}$ and $\mathbb{1}_z^{-\infty}(\zeta) = -\mathbb{1}_{z<\zeta}$, and the process yields half-plane supported entropy-entropy flux pairs.

Proposition 3.1. *(i) Let $H = H(w,z;\xi)$, $Q = Q(w,z;\xi)$ be C^1 functions. Then,*

$$\left\{ \begin{array}{l} Q_w = \lambda_1 H_w \ , \\ Q_z = \lambda_2 H_z \ , \end{array} \right. \qquad \textit{and} \quad Q = \lambda_1 H \ \textit{at} \ w = \xi, \tag{3.11}$$

is equivalent to

$$\eta = H(w,z;\xi)\, \mathbb{1}_w(\xi)\,, \quad q = Q(w,z;\xi)\, \mathbb{1}_w(\xi) \qquad \textit{satisfies (3.5) in } \mathcal{D}', \tag{3.12}$$

(i.e. (η, q) is a singular entropy pair).
(ii) Let $\bar H = \bar H(w,z;\zeta)$, $\bar Q = \bar Q(w,z;\zeta)$ be C^1 functions. Then,

$$\left\{ \begin{array}{l} \bar Q_w = \lambda_1 \bar H_w \ , \\ \bar Q_z = \lambda_2 \bar H_z \ , \end{array} \right. \qquad \textit{and} \quad \bar Q = \lambda_2 \bar H \ \textit{at} \ z = \zeta \quad , \tag{3.13}$$

is equivalent to

$$\bar\eta = \bar H(w,z;\zeta)\, \mathbb{1}_z(\zeta)\,, \quad \bar q = \bar Q(w,z;\zeta)\, \mathbb{1}_z(\zeta) \qquad \textit{satisfies (3.5) in } \mathcal{D}'. \tag{3.14}$$

Proof. The proof follows from direct computation. The statement (3.12) can be written,

$$\begin{aligned} q_w - \lambda_1 \eta_w &= (Q_w - \lambda_1 H_w)\mathbb{1}_w + (Q - \lambda_1 H)\delta(w - \xi) = 0\,, \\ q_z - \lambda_2 \eta_z &= (Q_z - \lambda_1 H_z)\mathbb{1}_w = 0\,, \end{aligned}$$

and thus the singular pair (η, q) solves (3.5) if and only if (3.11) holds. This shows (i); the proof of (ii) is similar. □

As explained above, the pairs (H, Q) in part (i) or $(\bar H, \bar Q)$ in part (ii) of Proposition 3.1 satisfy the same linear hyperbolic system

$$\left\{ \begin{array}{l} Q_w = \lambda_1 H_w \\ Q_z = \lambda_2 H_z\,, \end{array} \right. \tag{3.15}$$

and H or $\bar H$ are sought as solutions of the second order equation

$$H_{wz} = \frac{g_z}{g} H_w + \frac{f_w}{f} H_z\,. \tag{3.16}$$

We now show that the condition in (3.11) and (3.13) can be stated in terms of a Goursat problem. In this setting, H and $\bar H$ are required to satisfy different boundary conditions. Proposition 3.2 summarizes the minimal conditions for their construction.

Proposition 3.2. *(i) The problem (3.11) is equivalent to solve*

$$
\begin{aligned}
H_{wz} &= \frac{g_z}{g} H_w + \frac{f_w}{f} H_z \\
H(w = \xi, z; \xi) &= \tau(\xi)\, g(\xi, z)\,,
\end{aligned}
\tag{3.17}
$$

for some multiplicative factor $\tau(\xi)$, with the entropy flux $Q = Q(w, z; \xi)$ defined by

$$
\begin{aligned}
Q(w, z; \xi) &= \lambda_1(\xi, z) H(\xi, z; \xi) + \int_\xi^w \lambda_1(x, z)\, H_w(x, z; \xi)\, dx \\
&= \lambda_1(w, z) H(w, z; \xi) - \int_\xi^w \frac{\partial \lambda_1}{\partial w}(x, z)\, H(x, z; \xi)\, dx\,.
\end{aligned}
\tag{3.18}
$$

Therefore, such a pair (H, Q) generates a singular entropy-entropy flux pair of the type (3.12).
(ii) The problem (3.13) is equivalent to solve

$$
\begin{aligned}
\bar{H}_{wz} &= \frac{g_z}{g} \bar{H}_w + \frac{f_w}{f} \bar{H}_z \\
\bar{H}(w, z = \zeta; \zeta) &= \bar{\tau}(\zeta)\, f(w, \zeta)\,,
\end{aligned}
\tag{3.19}
$$

for some multiplicative factor $\bar{\tau}(\zeta)$, with $\bar{Q} = \bar{Q}(w, z; \zeta)$ defined by

$$
\begin{aligned}
\bar{Q}(w, z; \zeta) &= \lambda_2(w, \zeta) \bar{H}(w, \zeta; \zeta) + \int_\zeta^z \lambda_2(w, y)\, \bar{H}_z(w, y; \zeta)\, dy \\
&= \lambda_2(w, z) \bar{H}(w, z; \zeta) - \int_\zeta^z \frac{\partial \lambda_2}{\partial z}(w, y)\, \bar{H}(w, y; \zeta)\, dy\,,
\end{aligned}
\tag{3.20}
$$

Therefore, such a pair $(\bar{H}, \bar{Q})$ generates a singular entropy-entropy flux pair of the type (3.14).

Proof. Because of the equivalence between the equations (3.15) and (3.16), for the point (i), it is enough to prove the equivalence of the conditions at $w = \xi$. Then, together with (3.15), the condition $Q = \lambda_1 H$ implies that

$$
Q_z = \big(\lambda_1 H\big)_z = \lambda_2 H_z \quad \text{at } w = \xi\ .
$$

Using (3.8), the resulting differential equation yields

$$
\frac{\partial}{\partial z}\Big(\frac{H}{g}\Big) = 0 \quad \text{at } w = \xi\ ,
$$

that is H is constructed by solving (3.17). The associated Q is obtained by integrating the exact system (3.15). By (3.11) and (3.17),

$$
Q(w, z; \xi) = \tau(\xi)\, \lambda_1(\xi, z)\, g(\xi, z) \quad \text{at } w = \xi
$$

and thus Q is computed via (3.18). The choice of the integration constant, also gives the equivalence. The second form in (3.18) simply follows from an integration by parts.

The same reasoning on $(\bar{H}, \bar{Q})$ leads to the result (ii). $\square$

Remark 3.3.

1. There are two available "degrees of freedom" in defining (H,Q) and $(\bar{H},\bar{Q})$: one degree of freedom corresponds to the selection of the factors $\tau(\xi)$ or $\bar{\tau}(\zeta)$, and another corresponds to specifying a second boundary condition.

2. Even the choice $\tau(\xi)=0$ (or $\bar{\tau}(\zeta)=0$) provides an entropy H (respectively $\bar{H}$) that generates a nontrivial singular entropy pair. The resulting pair (η,q) is continuous but has jump discontinuities in the first derivatives along the line $w=\xi$ (or $z=\zeta$). This type of pairs is used in Serre [Se$_1$] to carry out the compensated compactness theory for systems of two conservation laws.

3. The system (3.10) admits singular solutions that involve delta masses. Consider the singular entropy pair $(\eta,q)=(H\mathbb{1}_w, Q\mathbb{1}_w)$ of the type (3.12). Then

$$\begin{cases} \eta_w = \varphi\ , \\ \eta_z = \psi\ , \end{cases} \qquad \begin{cases} q_w = \lambda_1\varphi\ , \\ q_z = \lambda_2\psi\ , \end{cases}$$

yields a distributional solution (φ,ψ) of (3.10):

$$\begin{aligned} \varphi &= \Phi(w,z;\xi)\mathbb{1}_w(\xi)\ +\ \tau(\xi)g(\xi,z)\delta(w-\xi)\ , \\ \psi &= \Psi(w,z;\xi)\mathbb{1}_w(\xi)\ , \end{aligned} \tag{3.21}$$

where $\Phi=H_w$, $\Psi=H_z$, and (Φ,Ψ) solve the Goursat problem

$$\begin{cases} \Phi_z = \frac{f_w}{f}\Psi + \frac{g_z}{g}\Phi\ , \\ \Psi_w = \frac{f_w}{f}\Psi + \frac{g_z}{g}\Phi\ , \end{cases} \qquad \text{with} \quad \Psi=\tau(\xi)g_z(\xi,z) \quad \text{at } w=\xi. \tag{3.22}$$

Similarly, the singular entropy pair $(\bar{\eta},\bar{q})=(\bar{H}\mathbb{1}_z,\bar{Q}\mathbb{1}_z)$ of the type (3.14) gives rise, through

$$\begin{cases} \bar{\eta}_w = \bar{\varphi}\ , \\ \bar{\eta}_z = \bar{\psi}\ , \end{cases} \qquad \begin{cases} \bar{q}_w = \lambda_1\bar{\varphi}\ , \\ \bar{q}_z = \lambda_2\bar{\psi}\ , \end{cases} \quad ,$$

to a distributional solution $(\bar{\varphi},\bar{\psi})$ of (3.10):

$$\begin{aligned} \bar{\varphi} &= \bar{\Phi}(w,z;\zeta)\mathbb{1}_z(\zeta)\ , \\ \bar{\psi} &= \bar{\Psi}(w,z;\zeta)\mathbb{1}_z(\zeta)\ +\ \bar{\tau}(\zeta)f(w,\zeta)\delta(z-\zeta)\ , \end{aligned} \tag{3.23}$$

where $\bar{\Phi}=\bar{H}_w$ and $\bar{\Psi}=\bar{H}_z$ satisfy the Goursat problem

$$\begin{cases} \bar{\Phi}_z = \frac{f_w}{f}\bar{\Psi} + \frac{g_z}{g}\bar{\Phi}\ , \\ \bar{\Psi}_w = \frac{f_w}{f}\bar{\Psi} + \frac{g_z}{g}\bar{\Phi}\ , \end{cases} \qquad \text{with} \quad \bar{\Phi}=\bar{\tau}(\zeta)f_w(w,\zeta) \quad \text{at } z=\zeta. \tag{3.24}$$

3.c The Goursat problem.

We now relate our construction to the classical Goursat problem

$$\eta_{wz} = \frac{g_z}{g}\eta_w + \frac{f_w}{f}\eta_z \,, \tag{3.25}$$

$$\begin{cases} \eta(w,0) = F(w) & \text{at } z=0\,, \\ \eta(0,z) = G(z) & \text{at } w=0\,, \end{cases} \tag{3.26}$$

where F and G are called the Goursat data. This problem is used to construct entropy pairs for systems of two conservation laws (see, *e.g.*, [La, Se_1, Da_2]). The existence theory is well understood in the strictly hyperbolic case. If λ_1 and λ_2 are C^1 and the data F, G are C^1 and compatible, $F(0)=G(0)$, there exists a unique solution η such that η, η_w, η_z and η_{wz} are continuous. If λ_1, λ_2 and F, G are C^2 then η is twice continously differentiable. Our goal is to prove a representation formula for solutions of the Goursat problem in terms of the fundamental kernels of Section 3.b. We proceed under the normalization $F(0)=G(0)=0$.

Let $(\mathcal{H}_1,\mathcal{Q}_1)$ and $(\mathcal{H}_2,\mathcal{Q}_2)$ be smooth entropy pairs defined as follows: The entropy $\mathcal{H}_1 = \mathcal{H}_1(w,z;\xi)$ is the solution of (3.17) subject to Goursat data

$$\begin{cases} \mathcal{H}_1 = \frac{g(\xi,z)}{g(\xi,0)} = \exp\big\{ -\int_0^z \frac{\lambda_{1z}}{\lambda_1-\lambda_2}(\xi,y)\,dy\big\} & \text{at } w=\xi\,, \\ \mathcal{H}_1 = 1 & \text{at } z=0\,, \end{cases} \tag{3.27}$$

$\mathcal{Q}_1 = \mathcal{Q}_1(w,z;\xi)$ is the associated entropy flux given by (3.18),

$$\mathcal{Q}_1(w,z;\xi) = \lambda_1(\xi,z)\frac{g(\xi,z)}{g(\xi,0)} + \int_\xi^w \lambda_1(x,z)\,\frac{\partial\mathcal{H}_1}{\partial w}(x,z;\xi)\,dx\,, \tag{3.28}$$

and satisfies

$$\begin{cases} \mathcal{Q}_1 = \lambda_1(\xi,z)\frac{g(\xi,z)}{g(\xi,0)} & \text{at } w=\xi\,, \\ \mathcal{Q}_1 = \lambda_1(\xi,0) & \text{at } z=0\,. \end{cases} \tag{3.29}$$

By Proposition 3.1, the pair $(\mathcal{H}_1,\mathcal{Q}_1)$ generates the singular entropy pair $(\mathcal{H}_1\mathbb{1}_w(\xi),\mathcal{Q}_1\mathbb{1}_w(\xi))$.

The entropy $\mathcal{H}_2 = \mathcal{H}_2(w,z;\zeta)$ is the solution of the Goursat problem (3.19) with data

$$\begin{cases} \mathcal{H}_2 = 1 & \text{at } w=0\,, \\ \mathcal{H}_2 = \frac{f(w,\zeta)}{f(0,\zeta)} = \exp\big\{ \int_0^w \frac{\lambda_{2w}}{\lambda_1-\lambda_2}(x,\zeta)\,dx\big\} & \text{at } z=\zeta\,, \end{cases} \tag{3.30}$$

$\mathcal{Q}_2 = \mathcal{Q}_2(w,z;\zeta)$ is the associated entropy flux, defined by

$$\mathcal{Q}_2(w,z;\zeta) = \lambda_2(w,\zeta)\frac{f(w,\zeta)}{f(0,\zeta)} + \int_\zeta^z \lambda_2(w,y)\,\frac{\partial\mathcal{H}_2}{\partial z}(w,y;\zeta)\,dy\,, \tag{3.31}$$

and satisfies

$$\begin{cases} \mathcal{Q}_2 = \lambda_2(0,\zeta) & \text{at } w=0\,, \\ \mathcal{Q}_2 = \lambda_2(w,\zeta)\frac{f(w,\zeta)}{f(0,\zeta)} & \text{at } z=\zeta\,. \end{cases} \tag{3.32}$$

The pair $(\mathcal{H}_2,\mathcal{Q}_2)$ generates the singular entropy pair $(\mathcal{H}_2\mathbb{1}_z(\zeta),\mathcal{Q}_2\mathbb{1}_z(\zeta))$.

We prove the representation theorem:

Theorem 3.4. *Let $(\mathcal{H}_1, \mathcal{Q}_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2)$ be defined in (3.27 - 3.32), and let the Goursat data be normalized so that $F(0) = G(0) = 0$. Then, the entropy entropy-flux pairs, solutions of the Goursat problem (3.25 - 3.26), are given by the representation formulas*

$$\eta(w,z) = \int_{\xi\in\mathbb{R}} \mathcal{H}_1(w,z;\xi)\mathbb{1}_w(\xi)\, F'(\xi)d\xi + \int_{\zeta\in\mathbb{R}} \mathcal{H}_2(w,z;\zeta)\mathbb{1}_z(\zeta)\, G'(\zeta)d\zeta\ , \tag{3.33}$$

$$q(w,z) = \int_{\xi\in\mathbb{R}} \mathcal{Q}_1(w,z;\xi)\mathbb{1}_w(\xi)\, F'(\xi)d\xi + \int_{\zeta\in\mathbb{R}} \mathcal{Q}_2(w,z;\zeta)\mathbb{1}_z(\zeta)\, G'(\zeta)d\zeta\ . \tag{3.34}$$

Proof. Let $(\mathcal{H}_1, \mathcal{Q}_1)$ satisfy (3.27 - 3.29). Consider the functions

$$\begin{aligned}
\eta_1(w,z) &= \int_{\xi\in\mathbb{R}} \mathcal{H}_1(w,z;\xi)\mathbb{1}_w(\xi)\, F'(\xi)d\xi\ ,\\
q_1(w,z) &= \int_{\xi\in\mathbb{R}} \mathcal{Q}_1(w,z;\xi)\mathbb{1}_w(\xi)\, F'(\xi)d\xi\ ,
\end{aligned}$$

Since $(\mathcal{H}_1\mathbb{1}_w, \mathcal{Q}_1\mathbb{1}_w)$ is a distributional solution of (3.5), it follows that (η_1, q_1) is an entropy-entropy flux pair. Moreover, (η_1, q_1) satisfies

$$\begin{aligned}
&\lim_{w\to 0} \eta_1(w,z) = 0\,, \qquad \lim_{w\to 0} q_1(w,z) = 0\,,\\
&\eta_1(w, z=0) = \int_{\xi\in\mathbb{R}} \mathcal{H}_1(w,0;\xi)\mathbb{1}_w(\xi)\, F'(\xi)d\xi = \int_{\xi\in\mathbb{R}} \mathbb{1}_w(\xi)\, F'(\xi)d\xi = F(w)\,,\\
&q_1(w, z=0) = \int_{\xi\in\mathbb{R}} \mathcal{Q}_1(w,0;\xi)\mathbb{1}_w(\xi)\, F'(\xi)d\xi = \int_{\xi\in\mathbb{R}} \lambda_1(\xi,0)\mathbb{1}_w(\xi)\, F'(\xi)d\xi\,.
\end{aligned}$$

Let $(\mathcal{H}_2, \mathcal{Q}_2)$ be defined by (3.30 - 3.32). Then (η_2, q_2), given by

$$\begin{aligned}
\eta_2(w,z) &= \int_{\zeta\in\mathbb{R}} \mathcal{H}_2(w,z;\zeta)\mathbb{1}_z(\zeta)\, G'(\zeta)d\zeta\\
q_2(w,z) &= \int_{\zeta\in\mathbb{R}} \mathcal{Q}_2(w,z;\zeta)\mathbb{1}_z(\zeta)\, G'(\zeta)d\zeta\,,
\end{aligned}$$

is an entropy-entropy flux pair that satisfies

$$\begin{aligned}
&\lim_{z\to 0} \eta_2(w,z) = 0\,, \qquad \eta_2(w=0,z) = \int_{\zeta\in\mathbb{R}} \mathbb{1}_z(\zeta)\, G'(\zeta)d\zeta = G(z)\,,\\
&\lim_{z\to 0} q_2(w,z) = 0, \qquad q_2(w=0,z) = \int_{\zeta\in\mathbb{R}} \lambda_2(0,\zeta)\mathbb{1}_z(\zeta)\, G'(\zeta)d\zeta\,.
\end{aligned}$$

The representation formula thus follows. ☐

3.d Half-plane and quarter-plane singular entropies.

In the above construction, the variables w and z do not play symmetric roles. It is natural to rather look for a comlete family of entropies where the symmetry is preserved. This family coincide with the choice of a single generating function H which can be used in both constructions (3.17), (3.19), i.e. that can be truncated in both w and z variables. As we will also see, a particular case of this construction yields the 'quarter plane' formulas obtained for isentropic gas dynamics in [LPT$_2$].

Let (ξ,ζ) be any fixed point in the plane, and define $H = H(w,z;\xi,\zeta)$ as the solution of (3.16) subject to Goursat data

$$\left\{\begin{array}{ll} H = \frac{g(\xi,z)}{g(\xi,\zeta)} = \exp\left\{ -\int_\zeta^z \frac{\lambda_{1z}}{\lambda_1-\lambda_2}(\xi,y)\,dy\right\} & \text{at } w=\xi\,, \\ H = \frac{f(w,\zeta)}{f(\xi,\zeta)} = \exp\left\{ \int_\xi^w \frac{\lambda_{2w}}{\lambda_1-\lambda_2}(x,\zeta)\,dx\right\} & \text{at } z=\zeta\,, \end{array}\right. \tag{3.35}$$

The entropy H is normalized by $H(\xi,\zeta;\xi,\zeta) = 1$. Let $Q = Q(w,z;\xi,\zeta)$ be the corresponding entropy flux, that satisfies the normalization condition $Q(\xi,\zeta;\xi,\zeta) = 0$, given by the formula

$$\begin{aligned} Q(w,z;\xi,\zeta) &= \lambda_1(\xi,z)H(\xi,z;\xi,\zeta) - \lambda_1(\xi,\zeta) + \int_\xi^w \lambda_1(x,z)H_w(x,z;\xi,\zeta)\,dx \\ &= \lambda_2(w,\zeta)H(w,\zeta;\xi,\zeta) - \lambda_2(\xi,\zeta) + \int_\zeta^z \lambda_2(w,y)H_z(w,y;\xi,\zeta)\,dy\,. \end{aligned} \tag{3.36}$$

Referring to Propositions 3.1 and 3.2, the entropy H generates singular entropy pairs of both types (3.12) and (3.14). The associated generators of the fluxes are (see (3.18), (3.20))

$$Q_1 = \lambda_1(\xi,\zeta) + Q(w,z;\xi,\zeta) \quad \text{and} \quad Q_2 = \lambda_2(\xi,\zeta) + Q(w,z;\xi,\zeta)$$

respectively, and differ by a constant. The pairs $(H, \lambda_1(\xi,\zeta)+Q)$ and $(H, \lambda_2(\xi,\zeta)+Q)$ generate the half-plane singular pairs

$$\eta_r = H\,\mathbb{1}_{\xi<w} \quad q_r = (\lambda_1(\xi,\zeta)+Q)\,\mathbb{1}_{\xi<w}\,, \tag{3.37}$$

$$\eta_l = H\,\mathbb{1}_{w<\xi} \quad q_l = (\lambda_1(\xi,\zeta)+Q)\,\mathbb{1}_{w<\xi}\,, \tag{3.38}$$

and

$$\eta_u = H\,\mathbb{1}_{\zeta<z} \quad q_u = (\lambda_2(\xi,\zeta)+Q)\,\mathbb{1}_{\zeta<z}\,, \tag{3.39}$$

$$\eta_d = H\,\mathbb{1}_{z<\zeta} \quad q_d = (\lambda_2(\xi,\zeta)+Q)\,\mathbb{1}_{z<\zeta}\,. \tag{3.40}$$

The pair (η_r, q_r) (resp. (η_l, q_l)) is supported on the right (resp. left) half plane $\{w>\xi\}$ (resp. $\{w<\xi\}$), the pair (η_u, q_u) (resp.(η_d, q_d)) is supported on the upper (resp. lower) half plane $\{z>\zeta\}$ (resp. $\{z<\zeta\}$).

Furthermore, by combining the above pairs, we produce the quarter-plane entropy pairs

$$
\eta_I = \eta_r + \eta_u - H = \begin{cases} H & \text{at } w > \xi,\, z > \zeta\,, \\ 0 & \text{at } w < \xi,\, z > \zeta\,, \\ -H & \text{at } w < \xi,\, z < \zeta\,, \\ 0 & \text{at } w > \xi,\, z < \zeta\,, \end{cases}
$$

$$
q_I = q_r + q_u - (\lambda_2(\xi,\zeta) + Q) = \begin{cases} \lambda_1(\xi,\zeta) + Q & \text{at } w > \xi,\, z > \zeta\,, \\ 0 & \text{at } w < \xi,\, z > \zeta\,, \\ -\lambda_2(\xi,\zeta) - Q & \text{at } w < \xi,\, z < \zeta\,, \\ (\lambda_1 - \lambda_2)(\xi,\zeta) & \text{at } w > \xi,\, z < \zeta\,, \end{cases} \tag{3.41}
$$

and

$$
\eta_{IV} = \eta_r - \eta_u = \begin{cases} 0 & \text{at } w > \xi,\, z > \zeta\,, \\ -H & \text{at } w < \xi,\, z > \zeta\,, \\ 0 & \text{at } w < \xi,\, z < \zeta\,, \\ H & \text{at } w > \xi,\, z < \zeta\,, \end{cases}
$$

$$
q_{IV} = q_r - q_u = \begin{cases} (\lambda_1 - \lambda_2)(\xi,\zeta) & \text{at } w > \xi,\, z > \zeta\,, \\ -\lambda_2(\xi,\zeta) - Q & \text{at } w < \xi,\, z > \zeta\,, \\ 0 & \text{at } w < \xi,\, z < \zeta\,, \\ \lambda_1(\xi,\zeta) + Q & \text{at } w > \xi,\, z < \zeta\,,. \end{cases} \tag{3.42}
$$

Note that η_I vanishes on the second and fourth quarters, while q_I vanishes on the second quarter and is constant on the fourth. Respectively, η_{IV} vanishes on the first and third quarters, while q_{IV} vanishes on the third quarter and is constant on the first. For isentropic gas dynamics, choosing $\xi = \zeta$, we have $(\lambda_1 - \lambda_2)(\xi,\zeta) = 0$. Also, the nonnegativity constraint on ρ, means that $w \le z$, and therefore the construction (3.42) gives the opposite of the kinetic function introduced in [LPT$_2$].

4. The equations of elastodynamics

Henceforth, we restrict attention to the equations describing isothermal, one-dimensional motions for elastic materials

$$
\begin{aligned} u_t - v_x &= 0\,, \\ v_t - \sigma(u)_x &= 0\,. \end{aligned} \tag{4.1}
$$

The function $\sigma(u)$ is assumed twice continuously differentiable with $\sigma'(u) > 0$. Then (4.1) is strictly hyperbolic with characteristic speeds

$$
\lambda_1(u) = -a(u)\,, \quad \lambda_2(u) = a(u)\,, \qquad \text{where } \; a(u) = \sqrt{\sigma'(u)}\,.
$$

The corresponding right and left eigenvectors, normalized with $l_i \cdot r_j = \delta_{ij}$ are given by

$$\begin{aligned} r_1 &= \frac{1}{2a(u)}(1, a(u)) \qquad & r_2 &= \frac{1}{2a(u)}(-1, a(u))\,, \\ l_1 &= (a(u), 1) & l_2 &= (-a(u), 1)\,. \end{aligned}$$

Entropy-entropy flux pairs for (4.1) are determined by solving the linear hyperbolic system

$$\begin{aligned} \eta_u + q_v &= 0 \\ a^2(u)\eta_v + q_u &= 0\,. \end{aligned} \tag{4.2}$$

The solvability condition for (4.2) is the linear wave equation

$$\eta_{uu} - a^2(u)\,\eta_{vv} = 0\,. \tag{4.3}$$

4.a Riemann invariants and entropies.

For the equations of elasticity the 1- and 2-Riemann invariants w and z are

$$w = v + A(u)\,, \quad z = v - A(u)\,, \qquad \text{where} \quad A(u) = \int_0^u a(s)ds\,. \tag{4.4}$$

The Riemann invariants define a mapping

$$T : (u, v) \in \mathbb{R}^2 \to (w, z) \in \{A(-\infty) < \frac{w - z}{2} < A(\infty)\}\,,$$

that is one-to-one and onto. If $\int_0^{\pm\infty} a(s)ds = \pm\infty$ then $T : \mathbb{R}^2 \to \mathbb{R}^2$. The inverse map is given by the formula

$$v = \frac{w + z}{2}\,, \quad u = A^{-1}\Big(\frac{w - z}{2}\Big)\,. \tag{4.5}$$

We note for future reference that $\nabla w = l_1$, $\nabla z = l_2$ and that

$$\partial_w = r_1 \cdot \nabla = \frac{1}{2a(u)}\partial_u + \frac{1}{2}\partial_v\,, \quad \partial_z = r_2 \cdot \nabla = -\frac{1}{2a(u)}\partial_u + \frac{1}{2}\partial_v\,.$$

The various quantities defined in Section 3 are expressed in the specific context of the elasticity equations. Set

$$b(w - z) := a \circ A^{-1}\Big(\frac{w - z}{2}\Big) = a(u)\,,$$

and note that

$$\lambda_1 = -a(u) = -b(w - z) < \lambda_2 = a(u) = b(w - z)\,.$$

It is convenient, fixing the integration constants, to write the solutions of (3.8b) in the form

$$
\begin{aligned}
f(w,z) &:= \exp\{\int_z^w \frac{\lambda_{2w}}{\lambda_1-\lambda_2}(x,z)dx\} &&= \exp\{-\int_0^{w-z}\frac{b'(s)}{2b(s)}ds\}\,,\\
g(w,z) &:= \exp\{-\int_w^z \frac{\lambda_{1z}}{\lambda_1-\lambda_2}(w,y)dy\} &&= \exp\{-\int_0^{w-z}\frac{b'(s)}{2b(s)}ds\}\,.
\end{aligned}
$$

Hence,

$$
f = g = e^{-M(w-z)}\,, \qquad \frac{g_z}{g} = -\frac{f_w}{f} = m(w-z)\,, \tag{4.6}
$$

where

$$
\begin{aligned}
b(\tau) &= a\circ A^{-1}\big(\frac{\tau}{2}\big)\,, & m(\tau) &= \frac{b'}{2b}(\tau) = \frac{a'}{4a^2}\circ A^{-1}\big(\frac{\tau}{2}\big)\,,\\
M(\tau) &= \int_0^\tau m(s)ds = \frac{1}{2}\ln\frac{b(\tau)}{b(0)}\,, & e^{-M(\tau)} &= \Big(\frac{b(0)}{b(\tau)}\Big)^{\frac12}\,.
\end{aligned}
$$

The equations determining the entropy-entropy flux pairs $\eta - q$, when expressed in the Riemann invariant domain, take the form

$$
\begin{aligned}
q_w &= -b(w-z)\eta_w\,,\\
q_z &= \ \ b(w-z)\eta_z\,,
\end{aligned} \tag{4.7}
$$

and (3.9) becomes

$$
\eta_{wz} = m(w-z)\eta_w - m(w-z)\eta_z\,. \tag{4.8}
$$

A second formulation of the problem incurs by introducing $\hat\varphi = \eta_w$ and $\hat\psi = \eta_z$; it yields the equivalent system

$$
\hat\varphi_z = \hat\psi_w = m(w-z)(\hat\varphi - \hat\psi)\,. \tag{4.9}
$$

4.b A specific construction of entropies for elastodynamics.

In the sequel, we use a formulation of the entropy-construction problem that is specific to the equations of elastodynamics. Given an entropy-entropy flux pair $\eta = \eta(u,v)$, $q = q(u,v)$, define

$$
\begin{aligned}
\varphi &= a(u)\eta - q\,,\\
\psi &= a(u)\eta + q\,.
\end{aligned} \tag{4.10}
$$

Then (η, q) is an entropy-entropy flux pair if and only if

$$
\begin{aligned}
\varphi_u - a(u)\varphi_v &= \frac{a'(u)}{2a(u)}(\varphi+\psi)\,,\\
\psi_u + a(u)\psi_v &= \frac{a'(u)}{2a(u)}(\varphi+\psi)\,,
\end{aligned} \tag{4.11}
$$

The latter may be expressed in the Riemann invariant domain, via the change of variables

$$\varphi(u,v) = \Phi(v + A(u), v - A(u))\,, \quad \psi(u,v) = \Psi(v + A(u), v - A(u))\,.$$

Then $(\Phi(w,z), \Psi(w,z))$ satisfies the system

$$\begin{aligned} \partial_z \Phi &= -m(w-z)(\Phi + \Psi)\,, \\ \partial_w \Psi &= \quad m(w-z)(\Phi + \Psi)\,, \end{aligned} \tag{4.12}$$

Conversely, given (Φ, Ψ) solution of (4.12), we define an entropy-entropy flux pair with the formulas

$$\begin{aligned} \eta &= \frac{1}{2a(u)}\big(\Phi + \Psi\big)(v + A(u), v - A(u))\,, \\ q &= -\frac{1}{2}\big(\Phi - \Psi\big)(v + A(u), v - A(u))\,. \end{aligned} \tag{4.13}$$

A variant of this formulation was introduced in Dafermos [Da_1] and another variant was used in Shearer [Sh].

The relevance of the formalism can be seen from the following lemma, due to Dafermos [Da_1].

Lemma 4.1. *Let $\eta - q$ be an entropy pair for the equations of elasticity, and φ, ψ be as in (4.10).*
(i) η is strictly convex at the point (u,v) if and only if $\varphi_{vv} > 0$, $\psi_{vv} > 0$ at (u,v).
(ii) The pairs (φ, ψ), (φ_v, ψ_v) and $(\varphi_{vv}, \psi_{vv})$ satisfy (4.11). Equivalently, $\eta_v - q_v$ and $\eta_{vv} - q_{vv}$ are entropy-entropy flux pairs.

Proof. Let φ, ψ be given by (4.10). Then

$$\begin{aligned} \varphi_{vv} &= a(u)\eta_{vv} - q_{vv} = a(u)\eta_{vv} + \eta_{vu}\,, \\ \psi_{vv} &= a(u)\eta_{vv} + q_{vv} = a(u)\eta_{vv} - \eta_{vu}\,, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \varphi_{vv}\,\psi_{vv} &= a^2(u)\eta_{vv}^2 - \eta_{vu}^2 = \eta_{uu}\eta_{vv} - \eta_{vu}^2 \\ \varphi_{vv} + \psi_{vv} &= 2a(u)\eta_{vv} \end{aligned}$$

Hence, part (i) follows. Part (ii) is due to the Galilean invariance of the equation (4.3) and the system (4.11) (or (4.12)) i.e. invariance under translations $v \to v + \alpha$ (equivalently $w \to w + \beta$, $z \to z + \beta$). $\square$

In the sequel, we use on occasion a hypothesis on the stress-strain response:

$$u\,\sigma''(u) > 0\,, \quad u \neq 0\,, \tag{c}$$

stating that $\sigma(u)$ is concave for $u < 0$ and convex for $u > 0$. Under hypothesis (c), the systems (4.11) and (4.12) enjoy certain maximum principles for the Cauchy and the Goursat problems. Consider the Cauchy problem with initial data at $u = 0$, the inflection point of $\sigma(u)$.

Lemma 4.2. *Let (φ, ψ) be a solution of (4.11), and suppose that (c) is satisfied.*

$$\text{If} \quad \begin{cases} \varphi = 0,\ \psi > 0 & \text{at } u = 0, \text{ or} \\ \varphi > 0,\ \psi = 0 & \text{at } u = 0, \text{ or} \\ \varphi,\ \psi > 0 & \text{at } u = 0 \end{cases} \qquad \text{then} \quad \varphi,\ \psi > 0 \text{ for } (u,v) \in \mathbb{R}^2 \text{ with } u \neq 0.$$

Proof. Let $\mathcal{C}$ be any "trapezoidal" domain that is bounded on the left side by a two-characteristic curve, on the right by a one-characteristic curve, on the bottom by the axis $u = 0$ and on the top by the line $u = \bar{u} > 0$. We prove that $\varphi + \psi > 0$ on $\bar{\mathcal{C}}$, the closure of $\mathcal{C}$. If this is not the case, let u_o be the first time that $\varphi + \psi$ vanishes on $\bar{\mathcal{C}}$; then $u_o > 0$. Let (u_o, v_o) be the point on $\bar{\mathcal{C}}$ where $\varphi + \psi$ first vanishes. Consider the backward characteristics emanating from (u_o, v_o) till they cut the axis $u = 0$. The coefficient $a'/2a$ in (4.11) is positive for $u > 0$. Integrating the equations along characteristics we see that $\varphi(u_o, v_o) > 0$ and $\psi(u_o, v_o) > 0$, which yields a contradiction. We conclude that $\varphi + \psi > 0$ on $\bar{\mathcal{C}}$ and, from (4.11), that $\varphi,\ \psi > 0$ on $\mathcal{C}$.

Let $\mathcal{C}'$ be now a domain bounded again by characteristics, the axis $u = 0$ on the top, and the line $u = \underline{u} < 0$ on the bottom. The coefficient $a'/2a$ is negative on $u < 0$, and a similar argument shows $\varphi,\ \psi > 0$ on $\mathcal{C}'$. □

Next, consider the Goursat problem

$$\begin{aligned} \partial_z \Phi &= -m(w - z)(\Phi + \Psi)\ , \\ \partial_w \Psi &= \quad m(w - z)(\Phi + \Psi)\ , \\ &\begin{cases} \Phi(w, z = 0) = \Phi_o(w)\ , \\ \Psi(w = 0, z) = \Psi_o(z)\ ,. \end{cases} \end{aligned} \tag{4.15}$$

Under hypothesis (c), solutions of the Goursat problem satisfy a maximum principle.

Lemma 4.3. *Let (Φ, Ψ) be a classical solution of (4.15), and suppose that (c) is satisfied.*

$$\text{If} \quad \begin{cases} \text{either} & \begin{cases} \Phi > 0 & \text{at } w > 0,\ z = 0 \\ \Psi = 0 & \text{at } w = 0,\ z < 0 \end{cases} \\ \text{or} & \begin{cases} \Phi = 0 & \text{at } w > 0,\ z = 0 \\ \Psi > 0 & \text{at } w = 0,\ z < 0 \end{cases} \end{cases} \qquad \text{then } \Phi, \Psi > 0 \ \text{ on } \{w > 0, z < 0\}.$$

$$\text{If} \quad \begin{cases} \text{either} & \begin{cases} \Phi < 0 & \text{at } w < 0,\ z = 0 \\ \Psi = 0 & \text{at } w = 0,\ z > 0 \end{cases} \\ \text{or} & \begin{cases} \Phi = 0 & \text{at } w < 0,\ z = 0 \\ \Psi < 0 & \text{at } w = 0,\ z > 0 \end{cases} \end{cases} \qquad \text{then } \Phi, \Psi < 0 \ \text{ on } \{w < 0, z > 0\}.$$

Proof. Observe that $m > 0$ on the fourth quadrant, $m < 0$ on the second quadrant, and $m = 0$ at the origin. We prove it for Goursat data satisfying

$$\Phi_o > 0 \quad \text{for } z = 0,\, w > 0, \qquad \Psi_o = 0 \quad \text{for } z < 0,\, w = 0.$$

First, we derive the result under the additional hypothesis $\Phi_o(0) > 0$. In this case (4.15) implies that

$$\begin{aligned} \Phi > 0\,, \quad \Psi > 0 \quad &\text{for } z = 0,\, w > 0, \\ \Phi > 0\,, \quad \Psi = 0 \quad &\text{for } z < 0,\, w = 0, \end{aligned}$$

and $\Phi + \Psi > 0$ at $(0,0)$. Let $\mathcal{C}$ be any domain of the form $\mathcal{C} = [0, \bar{w}] \times [\underline{z}, 0]$, with $\bar{w} > 0$ and $\underline{z} < 0$. Using an argument as in Lemma 4.2, we deduce $\Phi + \Psi > 0$ on $\mathcal{C}$. The system (4.12) then yields Φ, $\Psi > 0$ on the fourth quadrant.

If Φ_o is such that $\Phi_o(0) = 0$, then the Goursat data are approximated by data as in the previous paragraph, and the resulting solution satisfies Φ, $\Psi \geq 0$ on the fourth quadrant. These inequalities are in turn improved, by using (4.12), to obtain the final Φ, $\Psi > 0$ on the fourth quadrant. The proof of the other statements is similar. □

An analogous maximum principle holds for the Goursat problem for (4.11). The coordinate axes in the $w - z$ domain tranform to the (forward and backward) light rays

$$\begin{aligned} \mathcal{R}_{1+} &= \{(u, -A(u)) \;\; u > 0\}\,, \qquad & \mathcal{R}_{2+} &= \{(u, A(u)) \;\; u > 0\}\,, \\ \mathcal{R}_{1-} &= \{(u, -A(u)) \;\; u < 0\}\,, \qquad & \mathcal{R}_{2-} &= \{(u, A(u)) \;\; u < 0\}\,, \end{aligned}$$

in the $u - v$ domain. As a direct implication of Lemma 4.3:

Corollary 4.4. *Let $(\varphi(u,v), \psi(u,v))$ be a solution of (4.11), and suppose that (c) is satisfied.*

$$\text{If} \left\{ \begin{array}{ll} \text{either} & \left\{ \begin{array}{ll} \psi = 0 & \text{on } \mathcal{R}_{1+} \\ \varphi > 0 & \text{on } \mathcal{R}_{2+} \end{array} \right. \\ \text{or} & \left\{ \begin{array}{ll} \psi > 0 & \text{on } \mathcal{R}_{1+} \\ \varphi = 0 & \text{on } \mathcal{R}_{2+} \end{array} \right. \end{array} \right. \qquad \text{then } \varphi, \psi > 0 \;\; \text{on } \mathcal{C}_+ = \bigcup_{u>0} \big(-A(u), A(u) \big).$$

$$\text{If} \left\{ \begin{array}{ll} \text{either} & \left\{ \begin{array}{ll} \psi = 0 & \text{on } \mathcal{R}_{1-} \\ \varphi < 0 & \text{on } \mathcal{R}_{2-} \end{array} \right. \\ \text{or} & \left\{ \begin{array}{ll} \psi < 0 & \text{on } \mathcal{R}_{1-} \\ \varphi = 0 & \text{on } \mathcal{R}_{2-} \end{array} \right. \end{array} \right. \qquad \text{then } \varphi, \psi < 0 \;\; \text{on } \mathcal{C}_- = \bigcup_{u<0} \big(A(u), -A(u) \big).$$

Remark 4.5.

1. The quantities $\hat\varphi = \partial_w \eta$, $\hat\psi = \partial_z \eta$ entering in (4.9) should not be confused with the quantities φ, ψ defined in (4.10). Their relationship is

$$\partial_v^2 \varphi = a(u)\eta_{vv} + \eta_{vu} = 2a(u)\partial_w \eta_v = 2a(u)\partial_v \hat\varphi\,,$$
$$\partial_v^2 \psi = a(u)\eta_{vv} - \eta_{vu} = 2a(u)\partial_z \eta_v = 2a(u)\partial_v \hat\psi\,.$$

In the derivation we used the fact that ∂_w and ∂_z commute with ∂_v.

2. Lemmas 4.1 and 4.2 imply that, under hypothesis (c), an entropy η is strictly convex if and only if it emerges from initial data satisfying $\varphi_{vv} > 0$ and $\psi_{vv} > 0$ at $u = 0$. It is this observation that leads to the selection of the initial value problems determining the generators of the convex entropies.

4.c The kinetic functions I.

In this section we construct the singular entropy pairs $\Theta - \mathcal{J}$ and $\bar\Theta - \bar{\mathcal{J}}$, which serve as an intermediate step in order to define the kinetic functions. The entropy Θ is sought as the solution of the initial value problem

$$\tag{4.16} \begin{aligned} &\Theta_{uu} = a^2(u)\Theta_{vv}\ , \\ &\begin{cases} \Theta(u=0,v) = \mathbb{1}_{v>0}\ , \\ \Theta_u(u=0,v) = a(0)\delta(v)\ , \end{cases} \end{aligned}$$

$\mathcal{J}$ is the associated entropy flux:

$$\tag{4.17} \mathcal{J}_u = -a^2(u)\,\Theta_v\,, \quad \mathcal{J}_v = -\Theta_u\,;$$

$\bar\Theta$ is the solution of the initial value problem

$$\tag{4.18} \begin{aligned} &\bar\Theta_{uu} = a^2(u)\bar\Theta_{vv}\ , \\ &\begin{cases} \bar\Theta(u=0,v) = \mathbb{1}_{v>0}\ , \\ \bar\Theta_u(u=0,v) = -a(0)\delta(v)\ , \end{cases} \end{aligned}$$

and $\bar{\mathcal{J}}$ is the associated flux: $\bar\Theta - \bar{\mathcal{J}}$ are connected through (4.17).

Proposition 4.6. *(i) The singular pair $\Theta - \mathcal{J}$ is given by the formulas*

$$\tag{4.19} \begin{aligned} \Theta &= H\,\mathbb{1}_{v+A(u)>0} + (1-H)\,\mathbb{1}_{v-A(u)>0}\ , \\ \mathcal{J} &= Q\,\mathbb{1}_{v+A(u)>0} + (-a(0)-Q)\,\mathbb{1}_{v-A(u)>0}\ , \end{aligned}$$

where $H = H(u,v)$ is the smooth solution of the Goursat problem

$$\tag{4.20} \begin{aligned} &H_{uu} = a^2(u)H_{vv}\ , \\ &\begin{cases} H(u,-A(u)) = \left(\frac{a(0)}{a(u)}\right)^{\frac12} & \text{on } w = v + A(u) = 0\ , \\ H(u,A(u)) = 1 & \text{on } z = v - A(u) = 0\ , \end{cases} \end{aligned}$$

and $Q = Q(u,v)$ is the corresponding flux and attains the Goursat data

$$\begin{cases} Q(u,-A(u)) = -\big(a(0)a(u)\big)^{\frac{1}{2}} & \text{on } w = v + A(u) = 0\,, \\ Q(u,A(u)) = -a(0) & \text{on } z = v - A(u) = 0\,, \end{cases} \tag{4.21}$$

(ii) The singular entropy $\bar\Theta - \bar{\mathcal{J}}$ is given by

$$\begin{aligned} \bar\Theta &= (1-\bar H)\,\mathbb{1}_{v+A(u)>0} + \bar H\,\mathbb{1}_{v-A(u)>0}\ , \\ \bar{\mathcal{J}} &= (a(0)-\bar Q)\,\mathbb{1}_{v+A(u)>0} + \bar Q\,\mathbb{1}_{v-A(u)>0}\ , \end{aligned} \tag{4.22}$$

where $\bar H = \bar H(u,v)$, $\bar Q = \bar Q(u,v)$ is a smooth entropy-entropy flux pair defined by

$$\bar H(u,v) = H(u,-v)\,, \quad \bar Q(u,v) = -Q(u,-v)\,, \tag{4.23}$$

and attaining the Goursat data

$$\begin{aligned} &\begin{cases} \bar H(u,-A(u)) = 1 & \text{on } w = v + A(u) = 0\,, \\ \bar H(u,A(u)) = \left(\frac{a(0)}{a(u)}\right)^{\frac{1}{2}} & \text{on } z = v - A(u) = 0\,, \end{cases} \\ &\begin{cases} \bar Q(u,-A(u)) = a(0) & \text{on } w = v + A(u) = 0\,, \\ \bar Q(u,A(u)) = \big(a(0)a(u)\big)^{\frac{1}{2}} & \text{on } z = v - A(u) = 0\,, \end{cases} \end{aligned} \tag{4.24}$$

Proof. In preparation, we recall Proposition 3.2 adapted to the equations of elasticity.
(1) If $\mathcal{H}_1(w,z)$ is a smooth solution of (4.8) that for some τ_1 satisfies

$$\mathcal{H}_1(w=0,z) = \tau_1 e^{-M(-z)} \tag{4.25}$$

and $\mathcal{Q}_1$ is defined by

$$\mathcal{Q}_1(w,z) = -\tau_1 b(-z) e^{-M(-z)} - \int_0^w b(x-z)(\partial_w \mathcal{H}_1)(x,z)\,dx\ , \tag{4.26}$$

then $\mathcal{H}_1\mathbb{1}_{w>0}$ and $\mathcal{Q}_1\mathbb{1}_{w>0}$ are a singular entropy pair.
(2) If $\mathcal{H}_2(w,z)$ is a smooth entropy that for some τ_2 satisfies

$$\mathcal{H}_2(w,z=0) = \tau_2 e^{-M(w)} \tag{4.27}$$

and $\mathcal{Q}_2$ is defined by

$$\mathcal{Q}_2(w,z) = \tau_2 b(w) e^{-M(w)} + \int_0^z b(w-y)(\partial_z \mathcal{H}_2)(w,y)\,dy\ , \tag{4.28}$$

then $\mathcal{H}_2\mathbb{1}_{z>0}$ and $\mathcal{Q}_2\mathbb{1}_{z>0}$ form a singular entropy pair.

The above pairs can be expressed in the $u-v$ domain to provide distributional solutions for (4.2) of the form $H_1 1\!\!1_{v+A(u)>0}$, $Q_1 1\!\!1_{v+A(u)>0}$, and $H_2 1\!\!1_{v-A(u)>0}$, $Q_2 1\!\!1_{v-A(u)>0}$, where

$$H_i(u,v) = \mathcal{H}_i(v+A(u), v-A(u))\,, \quad Q_i(u,v) = \mathcal{Q}_i(v+A(u), v-A(u))\,, \quad i=1,2.$$

Consider a solution of (4.2) in the form of the linear combination

$$\Theta = H_1 1\!\!1_{v+A(u)>0} + H_2 1\!\!1_{v-A(u)>0}\,, \quad \mathcal{J} = Q_1 1\!\!1_{v+A(u)>0} + Q_2 1\!\!1_{v-A(u)>0}\,. \tag{4.29}$$

A computation shows that

$$\begin{aligned} \Theta(u=0,v) &= (H_1+H_2)\Big|_{u=0,v} 1\!\!1_{v>0} \\ \Theta_u(u=0,v) &= \big(\partial_u(H_1+H_2)\big)\Big|_{u=0,v} 1\!\!1_{v>0} + H_1(0,v)a(0)\delta(v) - H_2(0,v)a(0)\delta(v) \\ &= \big(\partial_u(H_1+H_2)\big)\Big|_{u=0,v} 1\!\!1_{v>0} + (\tau_1-\tau_2)a(0)\delta(v) \end{aligned} \tag{4.30}$$

with $\tau_1 = H_1(0,0)$, $\tau_2 = H_2(0,0)$.

Step 1. Construction of $\Theta - \mathcal{J}$

First, we turn to the solution of the initial value problem (4.16). If we select $\tau_1 = 1$, $\tau_2 = 0$ and the entropies H_1, H_2 so that (4.25), (4.27) and

$$(H_1+H_2)(0,v) = 1\,, \quad \partial_u(H_1+H_2)(0,v) = 0\,, \tag{4.31}$$

are satisfied, then Θ in (4.29) solves (4.16). Note that H_1+H_2 satisfies the initial value problem (4.3) with data (4.31). Hence, by uniqueness,

$$H_1 + H_2 = 1\,, \qquad \text{for all } u, v.$$

We turn to the construction of H_1, H_2. Let $\mathcal{H}(w,z)$ be the solution of the Goursat problem

$$\begin{aligned} &\mathcal{H}_{wz} = m(w-z)\mathcal{H}_w - m(w-z)\mathcal{H}_z\,, \\ &\begin{cases} \mathcal{H}(0,z) = e^{-M(-z)}\,, \\ \mathcal{H}(w,0) = 1\,, \end{cases} \end{aligned} \tag{4.32}$$

and let $\mathcal{Q}$ be the associated flux determined by

$$\begin{cases} \mathcal{Q}_w = -b(w-z)\mathcal{H}_w\,, \\ \mathcal{Q}_z = b(w-z)\mathcal{H}_z\,, \end{cases} \qquad \mathcal{Q}(0,0) = \lambda_1(0,0) = -b(0)\,. \tag{4.33}$$

Integrating (4.33), and using (4.32), (3.8) and(4.6), it follows that $\mathcal{Q}$ is given by the formulas

$$\begin{aligned} \mathcal{Q}(w,z) &= -b(-z)e^{-M(-z)} - \int_0^w b(x-z)\mathcal{H}_w(x,z)dx \\ &= -b(0) + \int_0^z b(w-y)\mathcal{H}_z(w,y)dy\,. \end{aligned} \tag{4.34}$$

The selections $\tau_1 = 1$, $\tau_2 = 0$ and $\mathcal{H}_1 = \mathcal{H}$, $\mathcal{H}_2 = 1 - \mathcal{H}$ meet all required conditions. By virtue of (4.26), (4.28) and (4.34), the associated fluxes are $\mathcal{Q}_1 = \mathcal{Q}$ and $\mathcal{Q}_2 = -b(0) - \mathcal{Q}$. The emerging singular pair $\Theta - \mathcal{J}$ is of the form (4.19), where

$$H(u,v) = \mathcal{H}(v + A(u), v - A(u))\,, \quad Q(u,v) = \mathcal{Q}(v + A(u), v - A(u))\,, \tag{4.35}$$

$H_1 = H$, $H_2 = 1 - H$, $Q_1 = Q$ and $Q_2 = -a(0) - Q$. The pair $H - Q$ is a smooth entropy pair that admits the Goursat data

$$\begin{aligned} H(u, -A(u)) &= \mathcal{H}(0, -2A(u)) = e^{-M(2A(u))} = \Big(\frac{a(0)}{a(u)}\Big)^{\frac{1}{2}}\ , \\ H(u, A(u)) &= \mathcal{H}(2A(u), 0) = 1\ , \\ Q(u, -A(u)) &= \mathcal{Q}(0, -2A(u)) = -b(2A(u))e^{-M(2A(u))} = -\big(a(0)a(u)\big)^{\frac{1}{2}}\ , \\ Q(u, A(u)) &= \mathcal{Q}(2A(u), 0) = -b(0) = -a(0)\ . \end{aligned}$$

Step 2. Construction of $\bar\Theta - \bar{\mathcal{J}}$

Next we take up the initial value problem (4.18). From (4.29), (4.30), we see that $\bar\Theta$ is a solution of (4.18), provided that $\tau_1 = 0$, $\tau_2 = 1$ and $\bar H_1$, $\bar H_2$ are smooth entropies satisfying (4.25), (4.27) and

$$(\bar H_1 + \bar H_2)(0, v) = 1\,, \quad \partial_u(\bar H_1 + \bar H_2)(0, v) = 0\,.$$

The latter implies

$$\bar H_1 + \bar H_2 = 1\,, \qquad \text{for all } u, v.$$

Let $\bar{\mathcal{H}}(w, z)$ be the solution of the Goursat problem

$$\begin{aligned} &\bar{\mathcal{H}}_{wz} = m(w-z)\bar{\mathcal{H}}_w - m(w-z)\bar{\mathcal{H}}_z\ , \\ &\begin{cases} \bar{\mathcal{H}}(0, z) = 1\ , \\ \bar{\mathcal{H}}(w, 0) = e^{-M(w)}\ , \end{cases} \end{aligned} \tag{4.36}$$

Then the selection $\tau_1 = 0$, $\tau_2 = 1$, $\bar{\mathcal{H}}_1 = 1 - \bar{\mathcal{H}}$, $\bar{\mathcal{H}}_2 = \bar{\mathcal{H}}$ meets all required conditions, and

$$\bar\Theta = \bar H_1 \mathbb{1}_{v + A(u) > 0} + \bar H_2 \mathbb{1}_{v - A(u) > 0}\,,$$

where $\bar H_1 = 1 - \bar H$, $\bar H_2 = \bar H$ and $\bar H(u, v) = \bar{\mathcal{H}}(v + A(u), v - A(u))$, solves (4.18).

Let $\bar{\mathcal{Q}}$ be the flux associated to $\bar{\mathcal{H}}$ and determined by

$$\begin{cases} \bar{\mathcal{Q}}_w = -b(w-z)\bar{\mathcal{H}}_w\ , \\ \bar{\mathcal{Q}}_z = b(w-z)\bar{\mathcal{H}}_z\ , \end{cases} \qquad \bar{\mathcal{Q}}(0,0) = \lambda_2(0,0) = b(0)\ . \tag{4.37}$$

Integrating the exact equation (4.37) and using (4.36), (3.8) and (4.6), we see that $\bar{\mathcal{Q}}$ is given by the formula

$$\begin{aligned}\bar{\mathcal{Q}}(w,z) &= b(0) - \int_0^w b(x-z)\bar{\mathcal{H}}_w(x,z)dx \\ &= b(w)e^{-M(w)} + \int_0^z b(w-y)\bar{\mathcal{H}}_z(w,y)dy\,.\end{aligned} \tag{4.38}$$

A comparison of (4.38) with (4.26) and (4.28) indicates that $\bar{\mathcal{Q}}_1 = b(0) - \bar{\mathcal{Q}}$, $\bar{\mathcal{Q}}_2 = \bar{\mathcal{Q}}$ and, thus, the flux $\bar{\mathcal{J}}$ associated to $\bar{\Theta}$ is

$$\bar{\mathcal{J}} = \bar{Q}_1 1\!\!1_{v+A(u)>0} + \bar{Q}_2 1\!\!1_{v-A(u)>0}\,,$$

where $\bar{Q}_1 = a(0) - \bar{Q}$, $\bar{Q}_2 = \bar{Q}$ and $\bar{Q}(u,v) = \bar{\mathcal{Q}}(v + A(u), v - A(u))$.

We conclude by noting that the Goursat problem (4.36) transforms to (4.32) via the change of variables $w \to -z$, $z \to -w$, $\bar{\mathcal{H}} \to \mathcal{H}$. Therefore,

$$\begin{aligned}&\bar{\mathcal{H}}(w,z) = \mathcal{H}(-z,-w)\,, \\ &\bar{H}(u,v) = \bar{\mathcal{H}}(v + A(u), v - A(u)) = \mathcal{H}(-v + A(u), -v - A(u)) = H(u,-v)\,.\end{aligned}$$

Moreover, (4.37) transforms to (4.33) via the change of variables $w \to -z$, $z \to -w$, $\bar{\mathcal{H}} \to \mathcal{H}$, $\bar{\mathcal{Q}} \to -\mathcal{Q}$. Accordingly,

$$\begin{aligned}&\bar{\mathcal{Q}}(w,z) = -\mathcal{Q}(-z,-w)\,, \\ &\bar{Q}(u,v) = \bar{\mathcal{Q}}(v + A(u), v - A(u)) = -\mathcal{Q}(-v + A(u), -v - A(u)) = -Q(u,-v)\,.\end{aligned}$$

This proves (4.23), and (4.24) follows from (4.20), (4.21) and (4.23). $\square$

The space $u - v$ is decomposed into four disjoint regions by the light rays $\mathcal{R}_1 = \mathcal{R}_{1+} \cup \mathcal{R}_{1-}$ and $\mathcal{R}_2 = \mathcal{R}_{2+} \cup \mathcal{R}_{2-}$ emanating from the origin. These are

$$\begin{aligned}\mathcal{C}_+ &:= \{u > 0\,,\ v \in (-A(u), A(u))\} \\ \text{left} &:= \{u > 0,\ v \in (-\infty, -A(u))\} \cup \{u < 0,\ v \in (-\infty, A(u))\} \\ \mathcal{C}_- &:= \{u < 0,\ v \in (A(u), -A(u))\} \\ \text{right} &:= \{u > 0,\ v \in (A(u), \infty)\} \cup \{u < 0,\ v \in (-A(u), \infty)\}\end{aligned} \tag{4.39}$$

where $\mathcal{C}_+$, $\mathcal{C}_-$ are the positive and negative light cones, respectively.

The construction in Proposition 4.6 is valid for general stress-strain laws. If the stress-strain law satisfies (c) then some remarkable properties connect the entropies $H_1 = H$, $H_2 = 1 - H$, $\bar{H}_1 = 1 - \bar{H}$, $\bar{H}_2 = \bar{H}$ with the associated entropy fluxes $Q_1 = Q$, $Q_2 = -a(0) - Q$, $\bar{Q}_1 = a(0) - \bar{Q}$, $\bar{Q}_2 = \bar{Q}$.

Lemma 4.7. *Under hypothesis (c),*

$$\begin{aligned} &0 < H < 1\,, \qquad 0 < \bar{H} < 1\,, \qquad \text{on } \mathcal{C}_+ \cup \mathcal{C}_-, \\ &|Q| < a(u)H\,, \quad |\bar{Q}| < a(u)\bar{H}\,, \quad \text{on } \mathcal{C}_+ \cup \mathcal{C}_-. \end{aligned} \tag{4.40}$$

The ratios Q_i/H_i are expressed as convex combinations of the wave speeds,

$$\begin{aligned} Q_1 &= \big[-a(u)\alpha_1 + a(u)(1-\alpha_1)\big]H_1\,, \quad Q_2 = \big[-a(u)\alpha_2 + a(u)(1-\alpha_2)\big]H_2\,, \\ \bar{Q}_1 &= \big[-a(u)\bar{\alpha}_1 + a(u)(1-\bar{\alpha}_1)\big]\bar{H}_1\,, \quad \bar{Q}_2 = \big[-a(u)\bar{\alpha}_2 + a(u)(1-\bar{\alpha}_2)\big]\bar{H}_2\,, \end{aligned} \tag{4.41}$$

where $\alpha_1 = \alpha_1(u,v)$ and $\alpha_2 = \alpha_2(u,v)$ satisfy the properties

$$0 < \alpha_1 < 1 \quad \text{on } \mathcal{C}_+ \cup \mathcal{C}_-\,, \qquad \begin{cases} \alpha_1 = 1 & \text{on } \mathcal{R}_1 \\ \alpha_1 = \frac{a(u)+a(0)}{2a(u)} & \text{on } \mathcal{R}_2 \end{cases}, \tag{4.42}$$

$$0 < \alpha_2 < 1 \quad \text{on } \mathcal{C}_+ \cup \mathcal{C}_-\,, \qquad \alpha_2 = \frac{\sqrt{a(u)} - \sqrt{a(0)}}{2\sqrt{a(u)}} \quad \text{on } \mathcal{R}_1, \tag{4.43a}$$

and, if $\sigma \in C^3$ with $\sigma'''(0) \neq 0$,

$$\alpha_2 = 0 \quad \text{on } \mathcal{R}_2\,; \tag{4.43b}$$

$\bar{\alpha}_1 = \bar{\alpha}_1(u,v)$ and $\bar{\alpha}_2 = \bar{\alpha}_2(u,v)$ are determined by

$$\bar{\alpha}_1(u,v) = 1 - \alpha_2(u,-v)\,, \qquad \bar{\alpha}_2(u,v) = 1 - \alpha_1(u,-v)\,. \tag{4.44}$$

Proof. For the entropy pair $H_1 - Q_1$, we introduce the decomposition

$$\varphi_1 = a(u)H_1 - Q_1 = a(u)H - Q\,, \quad \psi_1 = a(u)H_1 + Q_1 = a(u)H + Q\,.$$

Then (φ_1, ψ_1) satisfy (4.11) and attain the Goursat data

$$\begin{aligned} &\varphi_1(u, -A(u)) = 2\big(a(0)a(u)\big)^{\frac{1}{2}}\,, \quad \varphi_1(u, A(u)) = a(u) + a(0)\,, \\ &\psi_1(u, -A(u)) = 0\,, \qquad\qquad\qquad \psi_1(u, A(u)) = a(u) - a(0)\,. \end{aligned}$$

Corollary 4.4 implies

$$\varphi_1,\, \psi_1 > 0 \quad \text{on } \mathcal{C}_+ \text{ and } \mathcal{C}_-,$$

and, from (4.10), we have

$$\begin{aligned} H = H_1 &= \frac{1}{2a(u)}(\varphi_1 + \psi_1)\,, \\ Q = Q_1 &= \frac{1}{2}(\psi_1 - \varphi_1) = -a(u)\frac{\varphi_1}{\varphi_1 + \psi_1}H_1 + a(u)\frac{\psi_1}{\varphi_1 + \psi_1}H_1\,. \end{aligned}$$

We conclude that $H > 0$ on $\mathcal{C}_+ \cup \mathcal{C}_-$ and that Q_1/H_1 is expressed as a convex combination of the wave speeds. The factor $\alpha_1 = \varphi_1/(\varphi_1 + \psi_1)$ satisfies $0 < \alpha_1 < 1$ on $\mathcal{C}_+ \cup \mathcal{C}_-$ and takes the boundary values (4.42).

Next, for the entropy pair $H_2 - Q_2$, we introduce the decomposition

$$\begin{aligned}
\varphi_2 &= a(u)H_2 - Q_2 = a(u) + a(0) - (a(u)H - Q)\,, \\
\psi_2 &= a(u)H_2 + Q_2 = a(u) - a(0) - (a(u)H + Q)\,.
\end{aligned}$$

Then (φ_2, ψ_2) satisfy (4.11) subject to Goursat data

$$\begin{aligned}
\varphi_2(u, -A(u)) &= \left(\sqrt{a(u)} - \sqrt{a(0)}\right)^2 , & \varphi_2(u, A(u)) &= 0\,, \\
\psi_2(u, -A(u)) &= a(u) - a(0)\,, & \psi_2(u, A(u)) &= 0\,.
\end{aligned}$$

In view of (c), $a(u) - a(0) > 0$ for $u \neq 0$. Corollary 4.4 implies

$$\varphi_2,\ \psi_2 > 0 \quad \text{on } \mathcal{C}_+ \text{ and } \mathcal{C}_-\,,$$

and, from the formulas

$$\begin{aligned}
1 - H = H_2 &= \frac{1}{2a(u)}(\varphi_2 + \psi_2)\,, \\
-a(0) - Q = Q_2 &= \frac{1}{2}(\psi_2 - \varphi_2) = -a(u)\frac{\varphi_2}{\varphi_2 + \psi_2}H_2 + a(u)\frac{\psi_2}{\varphi_2 + \psi_2}H_2\,,
\end{aligned}$$

we deduce that $1 - H > 0$ on $\mathcal{C}_+ \cup \mathcal{C}_-$ and that Q_2/H_2 is written as a convex combination of the wave speeds.

The factor $\alpha_2 = \varphi_2/(\varphi_2 + \psi_2)$ satisfies $0 < \alpha_2 < 1$ on $\mathcal{C}_+ \cup \mathcal{C}_-$ and takes the value

$$\alpha_2(u, -A(u)) = \frac{\sqrt{a(u)} - \sqrt{a(0)}}{2\sqrt{a(u)}} \quad \text{on } \mathcal{R}_1\,.$$

To calculate $\alpha_2(u, A(u))$ we compute the limit $\lim_{z\to 0} \dfrac{\mathcal{Q}_2(w,z)}{\mathcal{H}_2(w,z)}$. This is an indeterminate limit as can be seen from (4.32), (4.34): $\mathcal{H}_2(w, 0) = 0$, $\mathcal{Q}_2(w, 0) = 0$. From (4.8) and (4.32), the function $\partial_z \mathcal{H}_2(w, 0)$ satisfies the equation

$$\partial_w\big(\partial_z \mathcal{H}_2(w,0)\big) + m(w)\big(\partial_z \mathcal{H}_2(w,0)\big) = 0\,, \tag{4.45}$$

and thus

$$\partial_z \mathcal{H}_2(w, 0) = [\partial_z \mathcal{H}_2(0,0)]\, e^{-M(w)}\,.$$

By (4.32), (4.7) and (c),

$$\partial_z \mathcal{H}_2(0,0) = \partial_z e^{-M(-z)}\big|_{z=0} = 0\,, \quad \partial_z \mathcal{H}_2(w,0) = 0\,, \quad \partial_z \mathcal{Q}_2(w,0) = 0\,.$$

Differentiating (4.8) we see that $\partial_{zz}\mathcal{H}_2(w,0)$ satisfies again (4.45) and

$$\partial_z \mathcal{H}_2(w,0) = [\partial_z \mathcal{H}_2(0,0)]\, e^{-M(w)}\,.$$

By (4.32) again,

$$\partial_{zz} \mathcal{H}_2(0,0) = \partial_{zz} e^{-M(-z)}\big|_{z=0} \neq 0\,,$$

by the assumption $\sigma'''(0) \neq 0$. Then

$$\lim_{z\to 0} \frac{\mathcal{Q}_2(w,z)}{\mathcal{H}_2(w,z)} = \lim_{z\to 0} \frac{\partial_z \mathcal{Q}_2(w,z)}{\partial_z \mathcal{H}_2(w,z)} = \lim_{z\to 0} \frac{\partial_{zz} \mathcal{Q}_2(w,z)}{\partial_{zz} \mathcal{H}_2(w,z)} = b(w)\,,$$

and (4.43b) follows. If $\sigma'''(0) = 0$, then a similar argument may be used to show that if $\sigma \in C^n$ and $\sigma^{(n)}(0) \neq 0$ then (4.43b) still holds. By contrast, in the case of linear elasticity the ratio Q_2/H_2 is indeterminate, see Remark 5.8.

Finally, we come to the pairs $\bar{H}_1 - \bar{Q}_1$ and $\bar{H}_2 - \bar{Q}_2$. The symmetry relations (4.23) imply that

$$\begin{aligned}
\bar{Q}_1(u,v) &= a(0) + Q(u,-v) = -Q_2(u,-v) \\
&= a(u)\alpha_2(u,-v) H_2(u,-v) - a(u)(1-\alpha_2(u,-v)) H_2(u,-v) \\
&= -a(u)(1-\alpha_2(u,-v)) \bar{H}_1(u,v) + a(u)\alpha_2(u,-v) \bar{H}_1(u,v)
\end{aligned}$$

and thus $\bar{Q}_1/\bar{H}_1$ can be written as a convex combination with factors determined by (4.44). The same is true for $\bar{Q}_2/\bar{H}_2$. ☐

Lemma 4.7 implies that under hypothesis (c) the flux $\mathcal{J}$ is expressed as

$$\mathcal{J} = \big(-a(u)\kappa + a(u)(1-\kappa)\big)\Theta\,.$$

This is seen from (4.19), (4.41) and from writing the constant state $-a(0)$ as

$$-a(0) = -a(u)\beta + a(u)(1-\beta)\,, \quad \text{with } \beta = \frac{a(u)+a(0)}{2a(u)} \in (0,1].$$

The function $\kappa = \kappa(u,v)$ takes values in $[0,1]$,

$$\kappa = \begin{cases} \frac{a(u)+a(0)}{2a(u)} & \text{on } \mathcal{R}_{2+} \\ \alpha_1 & \text{on } \mathcal{C}_+ \\ 1 & \text{on } \mathcal{R}_{1+} \\ 0 & \text{on left} \end{cases} \quad \text{and} \quad \kappa = \begin{cases} 0 & \text{on } \mathcal{R}_{1-} \\ \alpha_2 & \text{on } \mathcal{C}_- \\ \frac{\sqrt{a(u)}-\sqrt{a(0)}}{2\sqrt{a(u)}} & \text{on } \mathcal{R}_{1-} \\ \frac{a(u)+a(0)}{2a(u)} & \text{on right} \end{cases}$$

κ is continuous across $\mathcal{R}_2$ but discontinuous across $\mathcal{R}_1$. The factors α_1 and α_2 are connected through

$$\alpha_1 H + \alpha_2 (1-H) = \beta = \frac{a(u)+a(0)}{2a(u)},$$

a formula obtained by expressing the flux $Q_2 = -a(0) - Q_1$ in two different ways, by means of (4.41), and comparing the outcomes.

The pairs $\Theta - \mathcal{J}$ and $\bar\Theta - \bar{\mathcal{J}}$ serve to introduce the following singular entropy pairs, called kinetic functions :

$$\begin{aligned} &\Theta_r = \Theta\,, \quad \mathcal{J}_r = \mathcal{J}\,, && \Theta_l = -1+\Theta\,, \quad \mathcal{J}_l = a(0)+\mathcal{J}\,, \\ &\bar\Theta_r = \bar\Theta\,, \quad \bar{\mathcal{J}}_r = \bar{\mathcal{J}}\,, && \bar\Theta_l = -1+\bar\Theta\,, \quad \bar{\mathcal{J}}_l = -a(0)+\bar{\mathcal{J}}\,. \end{aligned} \tag{4.46}$$

The pair $\Theta_r - \mathcal{J}_r$ is defined in (4.19), $\Theta_l - \mathcal{J}_l$ reads

$$\begin{aligned} \Theta_l &= -H(u,v)\mathbb{1}_{v+A(u)<0} - \big(1-H(u,v)\big)\mathbb{1}_{v-A(u)<0}\,, \\ \mathcal{J}_l &= -Q(u,v)\mathbb{1}_{v+A(u)<0} - \big(-a(0)-Q(u,v)\big)\mathbb{1}_{v-A(u)<0}\,, \end{aligned} \tag{4.47}$$

$\bar\Theta_r - \bar{\mathcal{J}}_r$ is defined in (4.22), while $\bar\Theta_l - \bar{\mathcal{J}}_l$ reads

$$\begin{aligned} \bar\Theta_l &= -\big(1-\bar H(u,v)\big)\mathbb{1}_{v+A(u)<0} - \bar H(u,v)\mathbb{1}_{v-A(u)<0}\,, \\ \bar{\mathcal{J}}_l &= -\big(a(0)-\bar Q(u,v)\big)\mathbb{1}_{v+A(u)<0} - \bar Q(u,v)\mathbb{1}_{v-A(u)<0}\,, \end{aligned} \tag{4.48}$$

All are defined in terms of a single smooth entropy pair $H - Q$, determined by the Goursat problem (4.20) and (4.21), *cf.* (4.23). Each singular pair is a distributional solution of (4.2) emanating from initial data

$$\begin{aligned} &\Theta_r(0,v) = \mathbb{1}_{v>0}\,, \quad \Theta_l(0,v) = -\mathbb{1}_{v<0}\,, \qquad \partial_u\Theta_r(0,v) = \partial_u\Theta_l(0,v) = \ a(0)\delta_v\,, \\ &\bar\Theta_r(0,v) = \mathbb{1}_{v>0}\,, \quad \bar\Theta_l(0,v) = -\mathbb{1}_{v<0}\,, \qquad \partial_u\bar\Theta_r(0,v) = \partial_u\bar\Theta_l(0,v) = -a(0)\delta_v\,. \end{aligned}$$

By Lemma 4.7 and hypothesis (c), the associated fluxes satisfy

$$\begin{aligned} &\mathcal{J}_r = \big(-a(u)\kappa_r + a(u)(1-\kappa_r)\big)\Theta_r\,, && \mathcal{J}_l = \big(-a(u)\kappa_l + a(u)(1-\kappa_l)\big)\Theta_l\,, \\ &\bar{\mathcal{J}}_r = \big(-a(u)\bar\kappa_r + a(u)(1-\bar\kappa_r)\big)\bar\Theta_r\,, && \bar{\mathcal{J}}_l = \big(-a(u)\bar\kappa_l + a(u)(1-\bar\kappa_l)\big)\bar\Theta_l\,, \end{aligned} \tag{4.49}$$

where $\kappa_r = \kappa_r(u,v)$, $\kappa_l = \kappa_l(u,v)$, $\bar\kappa_r = \bar\kappa_r(u,v)$, $\bar\kappa_l = \bar\kappa_l(u,v)$ take values in $[0,1]$, satisfy

$$\bar\kappa_r(u,v) = 1-\kappa_l(u,-v)\,, \quad \bar\kappa_l(u,v) = 1-\kappa_r(u,-v)\,,$$

and are determined from α_1, α_2, $\bar\alpha_1$ and $\bar\alpha_2$ in (4.42 - 4.44).

The specific values of the kinetic functions are

$$(4.50)\quad \Theta_r = \begin{cases} H \\ 0 \\ 1-H \\ 1 \end{cases} \qquad \mathcal{J}_r = \begin{cases} Q & \text{on } \mathcal{C}_+ \\ 0 & \text{on left} \\ -a(0)-Q & \text{on } \mathcal{C}_- \\ -a(0) & \text{on right} \end{cases} \qquad \kappa_r = \begin{cases} \alpha_1 & \text{on } \mathcal{C}_+ \\ 1 & \text{on } \mathcal{R}_{1+} \\ 0 & \text{on left} \\ \alpha_2 & \text{on } \mathcal{C}_- \\ \frac{\sqrt{a(u)}-\sqrt{a(0)}}{2\sqrt{a(u)}} & \text{on } \mathcal{R}_{1-} \\ \frac{a(u)+a(0)}{2a(u)} & \text{on right} \end{cases}$$

$$(4.51)\quad \Theta_l = \begin{cases} -(1-H) \\ -1 \\ -H \\ 0 \end{cases} \qquad \mathcal{J}_l = \begin{cases} a(0)+Q & \text{on } \mathcal{C}_+ \\ a(0) & \text{on left} \\ -Q & \text{on } \mathcal{C}_- \\ 0 & \text{on right} \end{cases} \qquad \kappa_l = \begin{cases} \alpha_2 & \text{on } \mathcal{C}_+ \\ \frac{\sqrt{a(u)}-\sqrt{a(0)}}{2\sqrt{a(u)}} & \text{on } \mathcal{R}_{1+} \\ \frac{a(u)+a(0)}{2a(u)} & \text{on left} \\ \alpha_1 & \text{on } \mathcal{C}_- \\ 1 & \text{on } \mathcal{R}_{1-} \\ 0 & \text{on right} \end{cases}$$

$$(4.52)\quad \bar{\Theta}_r = \begin{cases} 1-\bar{H} \\ 0 \\ \bar{H} \\ 1 \end{cases} \qquad \bar{\mathcal{J}}_r = \begin{cases} a(0)-\bar{Q} & \text{on } \mathcal{C}_+ \\ 0 & \text{on left} \\ \bar{Q} & \text{on } \mathcal{C}_- \\ a(0) & \text{on right} \end{cases} \qquad \bar{\kappa}_r = \begin{cases} \frac{\sqrt{a(u)}+\sqrt{a(0)}}{2\sqrt{a(u)}} & \text{on } \mathcal{R}_{2+} \\ \bar{\alpha}_1 & \text{on } \mathcal{C}_+ \\ 1 & \text{on left} \\ 0 & \text{on } \mathcal{R}_{2-} \\ \bar{\alpha}_2 & \text{on } \mathcal{C}_- \\ \frac{a(u)-a(0)}{2a(u)} & \text{on right} \end{cases}$$

$$(4.53)\quad \bar{\Theta}_l = \begin{cases} -\bar{H} \\ -1 \\ -(1-\bar{H}) \\ 0 \end{cases} \qquad \bar{\mathcal{J}}_l = \begin{cases} -\bar{Q} & \text{on } \mathcal{C}_+ \\ -a(0) & \text{on left} \\ -a(0)+\bar{Q} & \text{on } \mathcal{C}_- \\ 0 & \text{on right} \end{cases} \qquad \bar{\kappa}_l = \begin{cases} 0 & \text{on } \mathcal{R}_{2+} \\ \bar{\alpha}_2 & \text{on } \mathcal{C}_+ \\ \frac{a(u)-a(0)}{2a(u)} & \text{on left} \\ \frac{\sqrt{a(u)}+\sqrt{a(0)}}{2\sqrt{a(u)}} & \text{on } \mathcal{R}_{2-} \\ \bar{\alpha}_1 & \text{on } \mathcal{C}_- \\ 1 & \text{on right} \end{cases}$$

The pairs $\Theta_r - \mathcal{J}_r$, $\Theta_l - \mathcal{J}_l$ and the factors κ_r, κ_l are discontinuous across $\mathcal{R}_1$ and continuous across $\mathcal{R}_2$. By contrast, the pairs $\bar{\Theta}_r - \bar{\mathcal{J}}_r$, $\bar{\Theta}_l - \bar{\mathcal{J}}_l$ and the factors $\bar{\kappa}_r$, $\bar{\kappa}_l$ are continuous across $\mathcal{R}_1$ and discontinuous across $\mathcal{R}_2$.

5. Kinetic formulation for elastodynamics

Consider the system of elastodynamics

$$\begin{aligned} u_t - v_x &= 0 \,, \\ v_t - \sigma(u)_x &= 0 \,, \end{aligned} \tag{5.1}$$

under the hypotheses $\sigma'(u) > 0$ and

$$u\,\sigma''(u) > 0 \quad u \neq 0 \,. \tag{c}$$

Entropy - entropy flux pairs $\eta(u,v) - q(u,v)$ for this system are constructed by solving (4.2), or equivalently the linear wave equation (4.3).

To ensure uniqueness of weak solutions, entropy inequalities need to be imposed. According to one premiss (motivated from zero-viscosity limits with identity diffusion matrices) all convex entropies must be dissipated, that is, weak solutions (u,v) in L^∞ are required to satisfy

$$\partial_t \eta(u,v) + \partial_x q(u,v) \leq 0 \qquad \text{in } \mathcal{D}' \,, \quad \text{for any convex entropy } \eta. \tag{5.2}$$

Our objective is to represent the convex entropies, under hypothesis (c), and to provide an equivalent definition for entropy weak solutions; this definition is called *kinetic formulation.* It represents by a set of two kinetic equations the full family of entropies, and also characterizes entirely those that are convex.

5.a The kinetic functions II.

We first introduce the kinetic functions $\Theta_o - \mathcal{J}_o$ and $\bar\Theta_o - \bar{\mathcal{J}}_o$, defined by

$$\begin{aligned} \Theta_o(u,v,\xi) &:= H(u,v-\xi)\,\mathbb{1}_{v+A(u)}(\xi) + \big(1 - H(u,v-\xi)\big)\,\mathbb{1}_{v-A(u)}(\xi) \\ &= \begin{cases} \Theta_r(u,v-\xi) & \xi > 0 \,, \\ \Theta_l(u,v-\xi) & \xi < 0 \,, \end{cases} \\ \mathcal{J}_o(u,v,\xi) &= Q(u,v-\xi)\,\mathbb{1}_{v+A(u)}(\xi) + \big(-a(0) - Q(u,v-\xi)\big)\,\mathbb{1}_{v-A(u)}(\xi) \\ &= \begin{cases} \mathcal{J}_r(u,v-\xi) & \xi > 0 \,, \\ \mathcal{J}_l(u,v-\xi) & \xi < 0 \,, \end{cases} \end{aligned} \tag{5.3}$$

$$\begin{aligned} \bar\Theta_o(u,v,\xi) &:= \big(1 - \bar H(u,v-\xi)\big)\mathbb{1}_{v+A(u)}(\xi) + \bar H(u,v-\xi)\mathbb{1}_{v-A(u)}(\xi) \\ &= \begin{cases} \bar\Theta_r(u,v-\xi) & \xi > 0 \,, \\ \bar\Theta_l(u,v-\xi) & \xi < 0 \,, \end{cases} \\ \bar{\mathcal{J}}_o(u,v,\xi) &= \big(a(0) - \bar Q(u,v-\xi)\big)\mathbb{1}_{v+A(u)}(\xi) + \bar Q(u,v-\xi)\mathbb{1}_{v-A(u)}(\xi) \\ &= \begin{cases} \bar{\mathcal{J}}_r(u,v-\xi) & \xi > 0 \,, \\ \bar{\mathcal{J}}_l(u,v-\xi) & \xi < 0 \,, \end{cases} \end{aligned} \tag{5.4}$$

The indicator functions are given by

$$\mathbb{1}_w(\xi) = \mathbb{1}(w,\xi) := \begin{cases} \mathbb{1}_{0<\xi<w} & \text{if } w > 0\,, \\ 0 & \text{if } w = 0\,, \\ -\mathbb{1}_{w<\xi<0} & \text{if } w < 0\,, \end{cases} \tag{5.5}$$

and satisfy the properties

$$\mathbb{1}_w(\xi) = \frac{1}{2}\big[\text{sign }(w-\xi) + \text{sign } \xi\big] \quad \text{a.e.,}$$
$$\partial_w \mathbb{1}_w(\xi) = \delta(w-\xi)\,, \quad \partial_\xi \mathbb{1}_w(\xi) = -\delta(w-\xi) + \delta(\xi)\,.$$

The entropy pairs $H-Q$ and $\bar{H}-\bar{Q}$ are smooth, they are determined by solving Goursat problems with boundary conditions

$$\begin{cases} H(u,-A(u)) = \left(\frac{a(0)}{a(u)}\right)^{\frac{1}{2}} \\ H(u,A(u)) = 1 \end{cases} \quad \text{and} \quad \begin{cases} \bar{H}(u,-A(u)) = 1 \\ \bar{H}(u,A(u)) = \left(\frac{a(0)}{a(u)}\right)^{\frac{1}{2}} \end{cases} \tag{5.6}$$

respectively (*cf.* Proposition 4.6), and satisfy the symmetry properties

$$\bar{H}(u,v) = H(u,-v)\,, \quad \bar{Q}(u,v) = -Q(u,-v)\,. \tag{5.7}$$

We refer to (4.46) - (4.53) for explicit formulas and further properties regarding $\Theta_o - \mathcal{J}_o$ and $\bar{\Theta}_o - \bar{\mathcal{J}}_o$.

We recall the notations

$$\mathcal{R}_1(\xi) = \mathcal{R}_{1+}(\xi) \cup \mathcal{R}_{1-}(\xi)\,, \quad \mathcal{R}_2(\xi) = \mathcal{R}_{2+}(\xi) \cup \mathcal{R}_{2-}(\xi)\,,$$

where $\mathcal{R}_{1+}(\xi)$ and $\mathcal{R}_{2+}(\xi)$ are the forward 1- and 2-characteristics emanating from ξ, $\mathcal{R}_{1-}(\xi)$ and $\mathcal{R}_{2-}(\xi)$ are the corresponding backward characteristics; also, $\mathcal{C}_+(\xi)$ and $\mathcal{C}_-(\xi)$ stand for the forward and backward light cones at ξ.

Proposition 5.1. *The kinetic functions enjoy the properties:*

(i) $\Theta_o - \mathcal{J}_o$ and $\bar{\Theta}_o - \bar{\mathcal{J}}_o$ are singular entropy pairs satisfying the initial-value problems

$$\begin{array}{ll} \Theta_{o,uu} = a^2(u)\Theta_{o,vv}\,, & \bar{\Theta}_{o,uu} = a^2(u)\bar{\Theta}_{o,vv}\,, \\ \begin{cases} \Theta_o(0,v,\xi) = \mathbb{1}_v(\xi)\,, \\ \partial_u\Theta_o(0,v,\xi) = a(0)\delta(v-\xi)\,, \end{cases} & \begin{cases} \bar{\Theta}_o(0,v,\xi) = \mathbb{1}_v(\xi)\,, \\ \partial_u\bar{\Theta}_o(0,v,\xi) = -a(0)\delta(v-\xi)\,. \end{cases} \end{array} \tag{5.8}$$

(ii) Θ_o, $\mathcal{J}_o$ have a jump discontinuity at $\mathcal{R}_1(\xi)$ and are continuous across $\mathcal{R}_2(\xi)$; $\bar{\Theta}_o$, $\bar{\mathcal{J}}_o$ are continuous across $\mathcal{R}_1(\xi)$ and have a jump discontinuity across $\mathcal{R}_2(\xi)$; they satisfy the symmetry properties

$$\begin{aligned} \bar{\Theta}_o(u,v,\xi) &= -\Theta_o(u,-v,-\xi)\,, \\ \bar{\mathcal{J}}_o(u,v,\xi) &= \mathcal{J}_o(u,-v,-\xi)\,, \end{aligned} \tag{5.9}$$

(iii) Θ_o, $\mathcal{J}_o$, $\bar\Theta_o$ *and* $\bar{\mathcal{J}}_o$ *take, for* $\xi \in \mathbb{R}$, *constant values outside the sets* $\mathcal{C}_+(\xi)$ *and* $\mathcal{C}_-(\xi)$, *and they are, for* (u,v) *fixed, of compact support in* ξ*:*

$$
\begin{aligned}
\text{supp } \Theta_o(u,v,\cdot) &= \text{supp } \mathcal{J}_o(u,v,\cdot) = \text{supp } \bar\Theta_o(u,v,\cdot) = \text{supp } \bar{\mathcal{J}}_o(u,v,\cdot) \\
&= \big[\min(v-|A(u)|,0)\,,\ \max(v+|A(u)|,0)\big]\,.
\end{aligned} \tag{5.10}
$$

(iv) Under hypothesis (c),

$$
\begin{aligned}
-1 \le \Theta_o(u,v,\xi) \le 0 \quad &\text{for } \xi<0, \qquad 0 \le \Theta_o(u,v,\xi) \le 1 \quad \text{for } \xi>0, \\
-1 \le \bar\Theta_o(u,v,\xi) \le 0 \quad &\text{for } \xi<0, \qquad 0 \le \bar\Theta_o(u,v,\xi) \le 1 \quad \text{for } \xi>0.
\end{aligned} \tag{5.11}
$$

The fluxes $\mathcal{J}_o$ *and* $\bar{\mathcal{J}}_o$ *are expressed as convex combinations of the wave speeds times the corresponding entropies*

$$
\begin{aligned}
\mathcal{J}_o &= \big(-a(u)\kappa_o + a(u)(1-\kappa_o)\big)\Theta_o\,, \\
\bar{\mathcal{J}}_o &= \big(-a(u)\bar\kappa_o + a(u)(1-\bar\kappa_o)\big)\bar\Theta_o\,,
\end{aligned} \tag{5.12}
$$

where

$$
\kappa_o(u,v,\xi) = \begin{cases} \kappa_r(u,v-\xi) & \xi>0 \\ \kappa_l(u,v-\xi) & \xi<0 \end{cases} \qquad \bar\kappa_o(u,v,\xi) = \begin{cases} \bar\kappa_r(u,v-\xi) & \xi>0 \\ \bar\kappa_l(u,v-\xi) & \xi<0 \end{cases}
$$

are defined in (4.50)-(4.53) and take values in $[0,1]$. *Also,* $\mathcal{J}_0$, $\bar{\mathcal{J}}_0$ *are bounded by*

$$
|\mathcal{J}_o| \le a(u)|\Theta_o|\,, \qquad |\bar{\mathcal{J}}_o| \le a(u)|\bar\Theta_o|\,. \tag{5.13}
$$

Proof. Part (i) is proved in Proposition 4.6. We verify directly the initial conditions:

$$
\begin{aligned}
\Theta_o(0,v,\xi) &= \mathbb{1}_v(\xi)\ , \\
\partial_u\Theta_o(0,v,\xi) &= H(0,v-\xi)a(0)\delta(v-\xi) - \big(1-H(0,v-\xi)\big)a(0)\delta(v-\xi) \\
&= a(0)\delta(v-\xi)\,,
\end{aligned}
$$

by (4.20). Similarly, by (4.24),

$$
\bar\Theta_o(0,v,\xi) = \mathbb{1}_v(\xi)\,, \quad \partial_u\bar\Theta_o(0,v,\xi) = -a(0)\delta(v-\xi)\,.
$$

Part (ii) follows from (5.7) and the property $\mathbb{1}_{-v}(-\xi) = -\mathbb{1}_v(\xi)$:

$$
\begin{aligned}
\bar\Theta_o(u,-v,-\xi) &= \big(1-\bar H(u,-v+\xi)\big)\mathbb{1}_{-v+A(u)}(-\xi) + \bar H(u,-v+\xi)\mathbb{1}_{-v-A(u)}(-\xi) \\
&= -\big(1-H(u,v-\xi)\big)\,\mathbb{1}_{v-A(u)}(\xi) - H(u,v-\xi)\,\mathbb{1}_{v+A(u)}(\xi) = -\Theta_o(u,v,\xi)\ , \\
\bar{\mathcal{J}}_o(u,-v,-\xi) &= \big(a(0)-\bar Q(u,-v+\xi)\big)\mathbb{1}_{-v+A(u)}(-\xi) + \bar Q(u,-v+\xi)\mathbb{1}_{-v-A(u)}(-\xi) \\
&= \big(-a(0)-Q(u,v-\xi)\big)\,\mathbb{1}_{v-A(u)}(\xi) + Q(u,v-\xi)\,\mathbb{1}_{v+A(u)}(\xi) = \mathcal{J}_o(u,v,\xi)\ .
\end{aligned}
$$

It reflects the fact that the first initial-value problem in (5.8) transforms to the second under the transformation of variables $u \to u$, $v \to -v$, $\xi \to -\xi$ and $\Theta \to -\Theta$.

Parts (iii) and (iv) follow from (4.50)-(4.53) in conjuction with Lemma 4.7 and (4.49). $\square$

It is seen from (4.46) that the derivatives $\partial_v \Theta_o$, $\partial_v \mathcal{J}_o$, $\partial_v \bar{\Theta}_o$ and $\partial_v \bar{\mathcal{J}}_o$ are all functions of $v - \xi$. This motivates to study the distributions

$$\begin{aligned} \mathcal{X} &= \partial_v \Theta\,, \quad \mathcal{Y} = \partial_v \mathcal{J}\,, \\ \bar{\mathcal{X}} &= \partial_v \bar{\Theta}\,, \quad \bar{\mathcal{Y}} = \partial_v \bar{\mathcal{J}}\,, \end{aligned} \tag{5.14}$$

where $\Theta - \mathcal{J}$ and $\bar{\Theta} - \bar{\mathcal{J}}$ are as in (4.16)-(4.18). Let also $\mathcal{F}$, $\mathcal{G}$, $\bar{\mathcal{F}}$, $\bar{\mathcal{G}}$ be defined by

$$\begin{aligned} \mathcal{F} &= a(u)\mathcal{X} - \mathcal{Y}\,, \quad \mathcal{G} = a(u)\mathcal{X} + \mathcal{Y}\,, \\ \bar{\mathcal{F}} &= a(u)\bar{\mathcal{X}} - \bar{\mathcal{Y}}\,, \quad \bar{\mathcal{G}} = a(u)\bar{\mathcal{X}} + \bar{\mathcal{Y}}\,. \end{aligned} \tag{5.15}$$

Then $\mathcal{X} - \mathcal{Y}$ and $\bar{\mathcal{X}} - \bar{\mathcal{Y}}$ satisfy the following properties.

Proposition 5.2. *The measures $\mathcal{X} - \mathcal{Y}$ and $\bar{\mathcal{X}} - \bar{\mathcal{Y}}$ are singular entropy pairs that enjoy the properties:*

(i) $\mathcal{X}$ and $\bar{\mathcal{X}}$ solve the problems

$$\begin{aligned} &\mathcal{X}_{uu} = a^2(u)\mathcal{X}_{vv}\,, && \bar{\mathcal{X}}_{uu} = a^2(u)\bar{\mathcal{X}}_{vv}\,, \\ &\begin{cases} \mathcal{X}(u=0,v) = \delta(v)\,, \\ \mathcal{X}_u(u=0,v) = a(0)\delta'(v)\,, \end{cases} && \begin{cases} \bar{\mathcal{X}}(u=0,v) = \delta(v)\,, \\ \bar{\mathcal{X}}_u(u=0,v) = -a(0)\delta'(v)\,, \end{cases} \end{aligned} \tag{5.16}$$

$\mathcal{Y}$ and $\bar{\mathcal{Y}}$ are the respective fluxes, and are given by the formulas

$$\begin{aligned} \mathcal{X} &= \Big(\frac{a(0)}{a(u)}\Big)^{\frac12}\delta(v + A(u)) + H_v\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,, \\ \mathcal{Y} &= -\big(a(0)a(u)\big)^{\frac12}\delta(v + A(u)) + Q_v\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \bar{\mathcal{X}} &= \Big(\frac{a(0)}{a(u)}\Big)^{\frac12}\delta(v - A(u)) - \bar{H}_v\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,, \\ \bar{\mathcal{Y}} &= \big(a(0)a(u)\big)^{\frac12}\delta(v - A(u)) - \bar{Q}_v\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,. \end{aligned} \tag{5.18}$$

(ii) $\mathcal{X}$, $\mathcal{Y}$, and $\bar{\mathcal{X}}$, $\bar{\mathcal{Y}}$ are of compact support in v

$$\text{supp } \mathcal{X}(u,\cdot) = \text{supp } \mathcal{Y}(u,\cdot) = \text{supp } \bar{\mathcal{X}}(u,\cdot) = \text{supp } \bar{\mathcal{Y}}(u,\cdot) = \begin{cases} [-A(u), A(u)] & u > 0 \\ [A(u), -A(u)] & u < 0 \end{cases}\,, \tag{5.19}$$

and satisfy the symmetry properties

$$\bar{\mathcal{X}}(u,v) = \mathcal{X}(u,-v)\,, \qquad \bar{\mathcal{Y}}(u,v) = -\mathcal{Y}(u,-v)\,. \tag{5.20}$$

(iii) $\mathcal{F}$, $\mathcal{G}$, $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$ are given by

$$
\begin{aligned}
\mathcal{F} &= 2\big(a(0)a(u)\big)^{\frac{1}{2}}\delta(v+A(u)) + (a(u)H_v - Q_v)\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,, \\
\mathcal{G} &= (a(u)H_v + Q_v)\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,,
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
\bar{\mathcal{F}} &= -\,(a(u)\bar{H}_v - \bar{Q}_v)\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,, \\
\bar{\mathcal{G}} &= 2\big(a(0)a(u)\big)^{\frac{1}{2}}\delta(v-A(u)) - (a(u)\bar{H}_v + \bar{Q}_v)\Big(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0}\Big)\,.
\end{aligned}
\tag{5.22}
$$

(iv) Under hypothesis (c),

$$
\begin{aligned}
&H_v > 0\,,\;\; \bar{H}_v < 0 \quad \textit{on } \mathcal{C}_+, \\
&H_v < 0\,,\;\; \bar{H}_v > 0 \quad \textit{on } \mathcal{C}_-,
\end{aligned}
\tag{5.23}
$$

and

$$
\begin{aligned}
&\mathcal{F} > 0\,, \quad \mathcal{G} > 0\,, \quad \mathcal{X} = \frac{1}{2a(u)}(\mathcal{F}+\mathcal{G}) > 0 \quad \textit{in } \mathcal{D}', \\
&\bar{\mathcal{F}} > 0\,, \quad \bar{\mathcal{G}} > 0\,, \quad \bar{\mathcal{X}} = \frac{1}{2a(u)}(\bar{\mathcal{F}}+\bar{\mathcal{G}}) > 0 \quad \textit{in } \mathcal{D}'.
\end{aligned}
\tag{5.24}
$$

Proof. Recalling that $\Theta - \mathcal{J}$ and $\bar{\Theta} - \bar{\mathcal{J}}$ satisfy the problems (4.16) - (4.18) and the invariance of (4.2) under the change of variables $v \to v + a$, we see that $\mathcal{X} - \mathcal{Y}$ and $\bar{\mathcal{X}} - \bar{\mathcal{Y}}$ are singular entropy pairs. The properties (i) and (iii) follow from a direct computation, (5.19) is a consequence of (5.17)-(5.18), while (5.20) follows from the property that the first initial value problem in (5.16) transforms into the second under the change of variable $v \to -v$.

In order to prove (iv), we consider, for the smooth entropy pair $H - Q$, the decomposition

$$
\varphi = a(u)H - Q\,, \quad \psi = a(u)H + Q\,.
$$

Then (φ, ψ) is a solution of (4.11) which, by (4.20 - 4.21), admits the Goursat data

$$
\begin{aligned}
&\varphi(u, -A(u)) = 2\big(a(0)a(u)\big)^{\frac{1}{2}}\,, \quad && \varphi(u, A(u)) = a(u) + a(0)\,, \\
&\psi(u, -A(u)) = 0\,, && \psi(u, A(u)) = a(u) - a(0)\,.
\end{aligned}
\tag{5.25}
$$

Differentiating the Goursat conditions, we see that

$$
\begin{aligned}
\psi_u(u, -A(u)) - a(u)\psi_v(u, -A(u)) &= 0\,, \\
\varphi_u(u, A(u)) + a(u)\varphi_v(u, A(u)) &= a'(u)\,,
\end{aligned}
$$

while from (4.11) and (5.25) we obtain

$$\varphi_u(u, A(u)) - a(u)\varphi_v(u, A(u)) = a'(u) \,,$$
$$\psi_u(u, -A(u)) + a(u)\psi_v(u, -A(u)) = a'(u)\Big(\frac{a(0)}{a(u)}\Big)^{\frac{1}{2}} .$$

In turn, the solution of the algebraic equations gives

$$\psi_v(u, -A(u)) = \frac{a'(u)}{2a(u)}\Big(\frac{a(0)}{a(u)}\Big)^{\frac{1}{2}}, \quad \varphi_v(u, A(u)) = 0 . \tag{5.26}$$

The function (φ_v, ψ_v) satisfies the Goursat problem (4.11),(5.26). Corollary 4.4 and hypothesis (c) then imply

$$\varphi_v,\ \psi_v > 0 \quad \text{on } \mathcal{C}_+ , \qquad \varphi_v,\ \psi_v < 0 \quad \text{on } \mathcal{C}_- .$$

Since

$$\varphi_v = a(u)H_v - Q_v \,, \quad \psi_v = a(u)H_v + Q_v \,,$$

and $(\mathbb{1}_{v+A(u)>0} - \mathbb{1}_{v-A(u)>0})$ equals $+1$ on $\mathcal{C}_+$ and -1 on $\mathcal{C}_-$, we conclude that $H_v > 0$ on $\mathcal{C}_+$, $H_v < 0$ on $\mathcal{C}_-$, and

$$\mathcal{F} > 0 \,, \quad \mathcal{G} > 0 \,, \quad \mathcal{X} = \frac{1}{2a(u)}(\mathcal{F} + \mathcal{G}) > 0 \quad \text{in } \mathcal{D}'.$$

The symmetry property $\bar{H}(u, v) = H(u, -v)$ and $\bar{Q}(u, v) = -Q(u, -v)$ yields

$$a(u)\bar{H}_v - \bar{Q}_v \,, \ \ a(u)\bar{H}_v + \bar{Q}_v \ \begin{cases} < 0 & \text{on } \mathcal{C}_+ \\ > 0 & \text{on } \mathcal{C}_- \end{cases} .$$

In turn, (5.18) and (5.22) imply

$$\bar{\mathcal{F}} > 0 \,, \quad \bar{\mathcal{G}} > 0 \,, \quad \bar{\mathcal{X}} = \frac{1}{2a(u)}(\bar{\mathcal{F}} + \bar{\mathcal{G}}) > 0 \quad \text{in } \mathcal{D}',$$

which completes the proof of part (iv). □

The singular entropy pair $\mathcal{X} - \mathcal{Y}$ encodes information on derivatives of the kinetic function $\Theta_o - \mathcal{J}_o$. This is seen via direct computation from (5.3), (5.4) and (4.46):

$$\begin{aligned} &\partial_v \Theta_o(u, v, \xi) = \mathcal{X}(u, v - \xi) \,, \quad \partial_\xi \Theta_o(u, v, \xi) = -\mathcal{X}(u, v - \xi) + \delta(\xi) \,, \\ &\partial_v \mathcal{J}_o(u, v, \xi) = \mathcal{Y}(u, v - \xi) \,, \quad \partial_\xi \mathcal{J}_o(u, v, \xi) \ = -\mathcal{Y}(u, v - \xi) - a(0)\delta(\xi) \,. \end{aligned} \tag{5.27}$$

A similar relation holds between $\bar{\mathcal{X}} - \bar{\mathcal{Y}}$ and $\bar{\Theta}_o - \bar{\mathcal{J}}_o$:

$$\begin{aligned} &\partial_v \bar{\Theta}_o(u, v, \xi) = \bar{\mathcal{X}}(u, v - \xi) \,, \quad \partial_\xi \bar{\Theta}_o(u, v, \xi) = -\bar{\mathcal{X}}(u, v - \xi) + \delta(\xi) \,, \\ &\partial_v \bar{\mathcal{J}}_o(u, v, \xi) = \bar{\mathcal{Y}}(u, v - \xi) \,, \quad \partial_\xi \bar{\mathcal{J}}_o(u, v, \xi) \ = -\bar{\mathcal{Y}}(u, v - \xi) + a(0)\delta(\xi) \,. \end{aligned} \tag{5.28}$$

5.b Representation of convex entropies.

The pairs $\Theta_o - \mathcal{J}_o$ and $\bar{\Theta}_o - \bar{\mathcal{J}}_o$ serve as generators of all entropy-entropy flux pairs, and at the same time, under hypothesis (c), as generators of the convex entropies. We have the following representation theorem.

Proposition 5.3. *Consider entropy - entropy flux pairs $\eta - q$ for the elasticity system, normalized so that $\eta(0,0) = q(0,0) = 0$. (i) They are represented by*

$$\begin{aligned} \eta(u,v) &= \int_{\mathbb{R}} \Theta_o(u,v,\xi)\partial_\xi p(\xi) d\xi + \int_{\mathbb{R}} \bar{\Theta}_o(u,v,\xi)\partial_\xi \bar{p}(\xi) d\xi \ , \\ q(u,v) &= \int_{\mathbb{R}} \mathcal{J}_o(u,v,\xi)\partial_\xi p(\xi) d\xi + \int_{\mathbb{R}} \bar{\mathcal{J}}_o(u,v,\xi)\partial_\xi \bar{p}(\xi) d\xi \ , \end{aligned} \tag{5.29}$$

where

$$\begin{aligned} p(v) &= \frac{1}{2}\eta(0,v) - \frac{1}{2a(0)} q(0,v) \ , \\ \bar{p}(v) &= \frac{1}{2}\eta(0,v) + \frac{1}{2a(0)} q(0,v) \ . \end{aligned} \tag{5.30}$$

(ii) They are also representated, using the notations in (5.16) - (5.18), by

$$\begin{aligned} \eta(u,v) &= \int_{\mathbb{R}} \mathcal{X}(u,v-\xi) p(\xi) d\xi + \int_{\mathbb{R}} \bar{\mathcal{X}}(u,v-\xi)\bar{p}(\xi) d\xi \ , \\ q(u,v) &= \int_{\mathbb{R}} \mathcal{Y}(u,v-\xi) p(\xi) d\xi + \int_{\mathbb{R}} \bar{\mathcal{Y}}(u,v-\xi)\bar{p}(\xi) d\xi \ , \end{aligned} \tag{5.31}$$

with p, $\bar{p}$ as in (5.30).

(iii) Under Hypothesis (c), such an entropy η is convex if and only if $p(\xi)$ and $\bar{p}(\xi)$ are convex.

The integrals in (5.31) are understood as pairings $< \cdot, \cdot >$ between the measures $\mathcal{X}$, $\bar{\mathcal{X}}$, $\mathcal{Y}$, $\bar{\mathcal{Y}}$ and test functions. The same convention is used whenever necessary in what follows. By contrast, the integrals in (5.29) are interpreted in the usual sense. This is one advantage of the use of $\Theta_o - \mathcal{J}_o$ and $\bar{\Theta}_o - \bar{\mathcal{J}}_o$.

Proof. Let $\eta - q$ be a given entropy pair, with $\eta(0,0) = q(0,0) = 0$, and set

$$A(v) = \eta(0,v) \,, \quad B(v) = \eta_u(0,v) = -q_v(0,v) \,.$$

Since Θ_o and $\bar{\Theta}_o$ satisfy (5.8), any function η of the form

$$\eta(u,v) = \int_{\mathbb{R}} \Theta_o(u,v,\xi)\partial_\xi p(\xi) d\xi + \int_{\mathbb{R}} \bar{\Theta}_o(u,v,\xi)\partial_\xi \bar{p}(\xi) d\xi$$

is an entropy which takes the initial data

$$\eta(0,v) = \int_{\mathbb{R}} 1\!\!1_v(\xi)\partial_\xi p(\xi)d\xi + \int_{\mathbb{R}} 1\!\!1_v(\xi)\partial_\xi \bar{p}(\xi)d\xi = (p(v)-p(0)) + (\bar{p}(v)-\bar{p}(0))\,,$$
$$\eta_u(0,v) = \int_{\mathbb{R}} a(0)\delta(v-\xi)\partial_\xi p(\xi)d\xi - \int_{\mathbb{R}} a(0)\delta(v-\xi)\partial_\xi \bar{p}(\xi)d\xi = a(0)p_v(v) - a(0)\bar{p}_v(v)\,.$$

If $p(v)$ and $\bar{p}(v)$ are selected as

$$\begin{aligned} p(v) &= \frac{1}{2}A(v) + \frac{1}{2a(0)}\int_0^v B(s)ds\,, \\ \bar{p}(v) &= \frac{1}{2}A(v) - \frac{1}{2a(0)}\int_0^v B(s)ds\,, \end{aligned} \tag{5.32}$$

then $\eta(0,v) = A(v)$ and $\eta_u(0,v) = B(v)$. For this selection it is $p(0) = \bar{p}(0) = 0$. The associated flux, normalized by $q(0,0) = 0$, is of the form

$$q(u,v) = \int_{\mathbb{R}} \mathcal{J}_o(u,v,\xi)\partial_\xi p(\xi)d\xi + \int_{\mathbb{R}} \bar{\mathcal{J}}_o(u,v,\xi)\partial_\xi \bar{p}(\xi)d\xi\,.$$

Using (5.27) and (5.28), the representation formulas can also be expressed as

$$\begin{aligned} q(u,v) &= < \mathcal{J}_o(u,v,\cdot), \partial_\xi p(\cdot) > + < \bar{\mathcal{J}}_o(u,v,\cdot), \partial_\xi \bar{p}(\cdot) > \\ &= < \mathcal{Y}(u,v-\cdot) + a(0)\delta(\cdot), p(\cdot) > + < \bar{\mathcal{Y}}(u,v-\cdot) - a(0)\delta(\cdot), \bar{p}(\cdot) > \\ &= \int_{\mathbb{R}} \mathcal{Y}(u,v-\xi)p(\xi)d\xi + \int_{\mathbb{R}} \bar{\mathcal{Y}}(u,v-\xi)\bar{p}(\xi)d\xi\,, \end{aligned}$$

where the last integrals over ξ denote the action of $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ on test functions, and we used that $p(0) = \bar{p}(0) = 0$.

Next we turn to part (iii). Using (5.27)-(5.29), we compute

$$\eta_{vv} = \int_{\mathbb{R}} \partial_v \Theta(u,v-\xi)p_{\xi\xi}(\xi)d\xi + \int_{\mathbb{R}} \partial_v \bar{\Theta}(u,v-\xi)\bar{p}_{\xi\xi}(\xi)d\xi\,, \tag{5.33}$$

and

$$\begin{aligned} \eta_{uu}\eta_{vv} - \eta_{vu}^2 &= \big(a(u)\eta_{vv} - q_{vv}\big)\big(a(u)\eta_{vv} + q_{vv}\big)\,, \\ &= \Big(\int_{\mathbb{R}} \mathcal{F}(u,v-\xi)p_{\xi\xi}(\xi) + \bar{\mathcal{F}}(u,v-\xi)\bar{p}_{\xi\xi}(\xi)d\xi\Big)\,, \\ &\qquad \Big(\int_{\mathbb{R}} \mathcal{G}(u,v-\xi)p_{\xi\xi}(\xi) + \bar{\mathcal{G}}(u,v-\xi)\bar{p}_{\xi\xi}(\xi)d\xi\Big)\,, \end{aligned} \tag{5.34}$$

where $\mathcal{F}$, $\mathcal{G}$, $\bar{\mathcal{F}}$, $\bar{\mathcal{G}}$ are the distributions defined in (5.15) (*cf.* Proposition 5.2). Since

$$\mathcal{F} > 0\,, \quad \mathcal{G} > 0\,, \quad \partial_v\Theta = \frac{1}{2a(u)}(\mathcal{F}+\mathcal{G}) > 0 \quad \text{in } \mathcal{D}',$$
$$\bar{\mathcal{F}} > 0\,, \quad \bar{\mathcal{G}} > 0\,, \quad \partial_v\bar{\Theta} = \frac{1}{2a(u)}(\bar{\mathcal{F}}+\bar{\mathcal{G}}) > 0 \quad \text{in } \mathcal{D}',$$

we deduce that, if p and $\bar p$ are convex, the resulting η is also convex. To see the converse note that

$$\mathcal{F}(0,v) = \bar{\mathcal{G}}(0,v) = 2a(0)\delta(v)\,, \quad \mathcal{G}(0,v) = \bar{\mathcal{F}}(0,v) = 0\,.$$

Evaluating (5.33) - (5.34) at $u=0$ we see that convexity of η implies

$$p_{vv} + \bar p_{vv} \geq 0\,, \quad p_{vv}\bar p_{vv} \geq 0\,,$$

that is p and $\bar p$ are both convex. □

An application of the Proposition 5.1 or of the representation formulas provides moment identities.

Corollary 5.4. *The kinetic functions satisfy the moment identities*

$$\begin{aligned} u &= \int_{\mathbb{R}} \frac{1}{2a(0)} (\Theta_o - \bar\Theta_o)(u,v,\xi)\,d\xi\,, \\ v &= -\int_{\mathbb{R}} \frac{1}{2a(0)} (\mathcal{J}_o - \bar{\mathcal{J}}_o)(u,v,\xi)\,d\xi\,, \end{aligned} \tag{5.35}$$

$$\begin{aligned} v &= \int_{\mathbb{R}} \frac{1}{2} (\Theta_o + \bar\Theta_o)(u,v,\xi)\,d\xi\,, \\ \sigma(u) - \sigma(0) &= -\int_{\mathbb{R}} \frac{1}{2} (\mathcal{J}_o + \bar{\mathcal{J}}_o)(u,v,\xi)\,d\xi\,, \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} \frac{1}{2}v^2 + W(u) := \frac{1}{2}v^2 + \int_0^u (\sigma(\tau) - \sigma(0))\,d\tau &= \int_{\mathbb{R}} \frac{\xi}{2}\,(\Theta_o + \bar\Theta_o)(u,v,\xi)\,d\xi\,, \\ v\big(\sigma(u) - \sigma(0)\big) &= -\int_{\mathbb{R}} \frac{\xi}{2}\,(\mathcal{J}_o + \bar{\mathcal{J}}_o)(u,v,\xi)\,d\xi\,, \end{aligned} \tag{5.37}$$

Proof. Taking $A(\xi) = 0$, $B(\xi) = 1$ in (5.29) and (5.32) we recover (5.35). The second identity is derived from the choice $A(\xi) = \xi$, $B(\xi) = 0$, and the last from the choice $A(\xi) = \xi^2/2$, $B(\xi) = \sigma(0)$ and (5.35). □

A similar argument yields the corresponding identities on $\mathcal{X}$, $\mathcal{Y}$.

Corollary 5.5. *The kinetic functions satisfy the moment identities*

$$\begin{aligned} u &= \int_{\mathbb{R}} \frac{\xi}{2a(0)} (\mathcal{X} - \bar{\mathcal{X}})(u,v-\xi)\,d\xi\,, \\ v &= -\int_{\mathbb{R}} \frac{\xi}{2a(0)} (\mathcal{Y} - \bar{\mathcal{Y}})(u,v-\xi)\,d\xi\,, \end{aligned} \tag{5.38}$$

$$
\begin{aligned}
v &= \int_{\mathbb{R}} \frac{\xi}{2} (\mathcal{X} + \bar{\mathcal{X}})(u, v - \xi)\, d\xi \ , \\
\sigma(u) - \sigma(0) &= - \int_{\mathbb{R}} \frac{\xi}{2} (\mathcal{Y} + \bar{\mathcal{Y}})(u, v - \xi)\, d\xi \ ,
\end{aligned}
\tag{5.39}
$$

and

$$
\begin{aligned}
\frac{1}{2} v^2 + W(u) &= \int_{\mathbb{R}} \frac{\xi^2}{4} \, (\mathcal{X} + \bar{\mathcal{X}})(u, v - \xi)\, d\xi \ , \\
v\big(\sigma(u) - \sigma(0)\big) &= - \int_{\mathbb{R}} \frac{\xi^2}{4} \, (\mathcal{Y} + \bar{\mathcal{Y}})(u, v - \xi)\, d\xi \ ,
\end{aligned}
\tag{5.40}
$$

5.c The kinetic formulation.

In this section we establish the equivalence between two definitions for weak solutions of the equations of elastodynamics. The first notion is the classical ''entropy" solution requiring the decrease of all convex entropies: (u, v) is a weak solution of (5.1) if it satisfies the decay property of convex entropies (5.2).

The second definition, called kinetic formulation, states that (u, v) is a weak solution of (5.1) if the kinetic functions $\Theta_o(u, v, \xi)$ and $\bar{\Theta}_o(u, v, \xi)$ satisfy for some positive, bounded measures m, $\bar{m}$ on $\mathbb{R}_x \times \mathbb{R}_t^+ \times \mathbb{R}_\xi$, the equations

$$
\begin{aligned}
\partial_t \Theta_o(u, v, \xi) + \partial_x \Big[\big(- a(u) \kappa_o + a(u)(1 - \kappa_o) \big) \Theta_o(u, v, \xi) \Big] &= \partial_\xi m(x, t, \xi) \ , \\
\partial_t \bar{\Theta}_o(u, v, \xi) + \partial_x \Big[\big(- a(u) \bar{\kappa}_o + a(u)(1 - \bar{\kappa}_o) \big) \bar{\Theta}_o(u, v, \xi) \Big] &= \partial_\xi \bar{m}(x, t, \xi) \ ,
\end{aligned}
\tag{5.41}
$$

in the sense of distributions in $\mathcal{D}'_{x,t,\xi}$.

Their precise relation depends on the smoothness class of the solution (u, v). This is expected, since the integrability of (u, v) reflects on the growth rates of entropy pairs that are used in (5.2). We state a theorem concerning L^∞ solutions and discuss in the remarks the extension to L^p solutions. The initial data are always taken in the sense of distributions.

Theorem 5.6. *Let $(u, v) \in L^\infty(\mathbb{R}_t^+ \times \mathbb{R}_x)$ have finite energy and suppose that (c) holds.*

(1) If $\Theta_o(u, v, \xi)$ and $\bar{\Theta}_o(u, v, \xi)$ satisfy (5.41) in $\mathcal{D}'_{x,t,\xi}$, for some positive bounded measures $m(x, t, \xi)$, $\bar{m}(x, t, \xi)$ on $\mathbb{R}_x \times \mathbb{R}_t^+ \times \mathbb{R}_\xi$, then (u, v) satisfy (5.2) for any C^2 entropy-entropy flux pair with η convex.

(2) If (u, v) satisfies the entropy inequalities (5.2) in $\mathcal{D}'_{x,t}$ for any C^2 entropy-entropy flux pairs $\eta - q$ with η convex, then there exist unique positive, bounded measures $m(x, t, \xi)$, $\bar{m}(x, t, \xi)$ on $\mathbb{R}_x \times \mathbb{R}_t^+ \times \mathbb{R}_\xi$ such that $\Theta_o(u, v, \xi)$ and $\bar{\Theta}_o(u, v, \xi)$ satisfy (5.41) in $\mathcal{D}'_{x,t,\xi}$.

(3) The measures m and $\bar{m}$ obey the bound

$$
\int_{\mathbb{R}_x \times \mathbb{R}_t^+ \times \mathbb{R}_\xi} m + \bar{m} \le \int_{\mathbb{R}_x} [\frac{1}{2} v_0^2(x) + W(u_0(x))] dx \ .
\tag{5.42}
$$

(4) Moreover, $m(t,x,\xi) = \bar{m}(t,x,\xi) = 0$ for $|\xi| > \| \, v + |A(u)| \, \|_{L^\infty}$, and thus test functions in (5.41) may grow at infinity in ξ. Also, when u, v are C^1 in some open set Ω of $\mathbb{R}_x \times \mathbb{R}_t^+$ then $m = \bar{m} = 0$ for $(x,t) \in \Omega$, $\xi \in \mathbb{R}$.

We begin with an estimate that connects the growth rate of test functions (in ξ) with the growth rate (in u, v) of the entropy pairs $\eta - q$.

Lemma 5.7. *Assume that (c) holds.*

(i) If the growth of $p(\xi) \in C^1(\mathbb{R}_\xi)$ is controlled by

$$|\partial_\xi p(\xi)| \le O(1)\big(1 + |\partial_\xi \Psi(\xi)|\big)\,, \quad \xi \in \mathbb{R} \tag{5.43}$$

for some convex function $\Psi(\xi)$ with $\Psi(0) = \Psi'(0) = 0$, then

$$\begin{aligned} \Big| \int_{\mathbb{R}_\xi} \Theta_o\, p_\xi d\xi \Big| &\le O(1)\big(|v + A(u)| + |v - A(u)| + \Psi(v + A(u)) + \Psi(v - A(u))\big) \\ \Big| \int_{\mathbb{R}_\xi} \mathcal{J}_o\, p_\xi d\xi \Big| &\le O(1) a(u)\big(|v + A(u)| + |v - A(u)| + \Psi(v + A(u)) + \Psi(v - A(u))\big) \end{aligned} \tag{5.44}$$

(ii) If for some $q \ge 0$

$$|\partial_\xi p(\xi)| \le O(1)\big(1 + |\xi|^q\big)\,, \quad \xi \in \mathbb{R} \tag{5.45}$$

then

$$\begin{aligned} \Big| \int_{\mathbb{R}_\xi} \Theta_o\, p_\xi d\xi \Big| &\le O(1)\big(|v + A(u)| + |v - A(u)| + |v + A(u)|^{q+1} + |v - A(u)|^{q+1}\big) \\ \Big| \int_{\mathbb{R}_\xi} \mathcal{J}_o\, p_\xi d\xi \Big| &\le O(1) a(u)\big(|v + A(u)| + |v - A(u)| + |v + A(u)|^{q+1} + |v - A(u)|^{q+1}\big) \end{aligned} \tag{5.46}$$

Proof. Let $\Psi(\xi)$ be a convex function with $\Psi(0) = \Psi'(0) = 0$. For $p(\xi) \in C^1(\mathbb{R}_\xi)$ with growth controlled by (5.43) we have, from (5.3) (or (5.4)), (4.40) and (2.7), the estimate

$$\begin{aligned} \int_{\mathbb{R}_\xi} |\Theta_o\, p_\xi| \, d\xi &\le O(1) \int_{\mathbb{R}_\xi} \big(|\mathbb{1}_{v+A(u)}(\xi)| + |\mathbb{1}_{v-A(u)}(\xi)|\big)\big(1 + |\partial_\xi \Psi(\xi)|\big) \, d\xi \\ &\le O(1)\big(|v + A(u)| + |v - A(u)| + \Psi(v + A(u)) + \Psi(v - A(u))\big) \end{aligned}$$

Moreover, from (5.13),

$$\int_{\mathbb{R}_\xi} |\mathcal{J}_o\, p_\xi| \, d\xi \le a(u) \int_{\mathbb{R}_\xi} |\Theta_o\, p_\xi| \, d\xi$$

and (5.44) follows. The proof of part (ii) is similar. $\square$

Proof of Theorem 5.6. We first consider (1). Let $\Theta_o(u,v,\xi) - \mathcal{J}_o(u,v,\xi)$, $\bar\Theta_o(u,v,\xi) - \bar{\mathcal{J}}_o(u,v,\xi)$ satisfy (5.41) for some positive, bounded measures $m(x,t,\xi)$, $\bar m(x,t,\xi)$ on $\mathbb{R}_x \times \mathbb{R}_t^+ \times \mathbb{R}_\xi$. For the first equation in (5.41) this means

$$
\begin{aligned}
&- \int_{\mathbb{R}_x \times [0,T]} \int_{\mathbb{R}_\xi} \Theta_o \, P_\xi \, \varphi_t + \mathcal{J}_o \, P_\xi \, \varphi_x d\xi \, dx \, dt \\
&\qquad\qquad - \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} \Theta_o(u_0(x), v_0(x), \xi) \, P_\xi(\xi) \, \varphi(x,0) d\xi \, dx \\
&\qquad\qquad = - \int_{\mathbb{R}_x \times [0,T]} \int_{\mathbb{R}_\xi} P_{\xi\xi}(\xi) \, \varphi(x,t) \, dm(x,t,\xi)
\end{aligned} \tag{5.47}
$$

for $\varphi \in C_c^\infty(\mathbb{R}_x \times [0,T))$, $P \in C_c^2(\mathbb{R}_\xi)$. An analogous identity is provided by the second equation. The result follows, formally, by testing these identities against convex test functions p and $\bar p$ and using the positivity of the measures and the representation fomulas (5.29). As P does not cover the class of convex test functions a validation is needed for the formal argument.

First, (5.47) is extended to hold for test functions $P \in C^2(\mathbb{R}_\xi)$ with $P_{\xi\xi}$ of compact support. To this end, let $g \in C^\infty(\mathbb{R}_\xi)$ with $g(\xi) = 1$ on $[-1,1]$ and supp $g \subset [-2,2]$. Define the test functions $P^n(\xi) = P(\xi)\, g(\xi/n)$ in $C_c^2(\mathbb{R}_\xi)$. Then

$$
\begin{aligned}
&\partial_\xi P^n(\xi) = \partial_\xi P(\xi) \, g(\frac{\xi}{n}) + P(\xi) \frac{1}{n} g'(\frac{\xi}{n}) \to \partial_\xi P(\xi) \,, \qquad \text{for } \xi \in \mathbb{R}. \\
&|\partial_\xi P^n| \le O(1)(1 + |\xi|)
\end{aligned}
$$

Since $(u,v) \in L^\infty_{x,t}$, Lemma 5.7 and the dominated convergence theorem allow to pass to the limit in the first three terms in (5.47). For the last term, the boundedness of m gives

$$
\int_{\mathbb{R}_x \times [0,T]} \int_{\mathbb{R}_\xi} P^n_{\xi\xi} \, \varphi dm \to \int_{\mathbb{R}_x \times [0,T]} \int_{\mathbb{R}_\xi} P_{\xi\xi} \, \varphi dm
$$

Testing (5.47) against $\varphi \ge 0$ and P convex (with $P_{\xi\xi}$ of compact support), we obtain

$$
\begin{aligned}
&- \int_{\mathbb{R}_x \times [0,T]} \int_{\mathbb{R}_\xi} \Theta_o \, P_\xi \, \varphi_t + \mathcal{J}_o \, P_\xi \, \varphi_x d\xi \, dx \, dt \\
&\qquad\qquad - \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} \Theta_o(u_0(x), v_0(x), \xi) \, P_\xi(\xi) \, \varphi(x,0) d\xi \, dx \le 0
\end{aligned} \tag{5.48}
$$

To extend (5.48) for arbitrary convex test functions p, define

$$
P^n(\xi) = p_0 + p_1 \xi + \int_0^\xi \int_0^\zeta p_{\xi\xi}(s) g(\frac{s}{n}) \, ds d\zeta
$$

where g is as before, $p_0 = p(0)$, $p_1 = p_\xi(0)$. Then $\partial_{\xi\xi} P^n$ is of compact support,

$$
\begin{aligned}
&\partial_\xi P^n \to \partial_\xi p \,, \quad \text{for } \xi \in \mathbb{R}, \\
&|\partial_\xi P^n| \le O(1)\big(1 + |\partial_\xi \Psi(\xi)|\big) \,, \qquad \text{where } \Psi(\xi) = p(\xi) - p_0 - p_1 \xi.
\end{aligned}
$$

Using part (i) of Lemma 5.7, (5.48) is extended for arbitrary convex test functions p. The result follows from the entropy representation formulas.

Next, we consider (2). Let $(u,v) \in L^\infty_{x,t}$ satisfy (5.2). First, we define the distributions m and $\bar{m}$ so that

$$\partial_t \Theta_o + \partial_x \mathcal{J}_o = \partial_\xi m\,, \quad \partial_t \bar{\Theta}_o + \partial_x \bar{\mathcal{J}}_o = \partial_\xi \bar{m}\,, \quad \text{in } \mathcal{D}'_{x,t,\xi}. \tag{5.49}$$

This is done as follows: By (5.46),

$$\begin{aligned} &\int_{\mathbb{R}_\xi} |\Theta_o| d\xi\ , \int_{\mathbb{R}_\xi} |\bar{\Theta}_o| d\xi \le O(1)\big(|v + A(u)| + |v - A(u)|\big) \\ &\int_{\mathbb{R}_\xi} |\mathcal{J}_o| d\xi\ , \int_{\mathbb{R}_\xi} |\bar{\mathcal{J}}_o| d\xi \le O(1) a(u)\big(|v + A(u)| + |v - A(u)|\big) \end{aligned} \tag{5.50}$$

and thus for $(u,v) \in L^\infty_{x,t}$ we have

$$\int_{-\infty}^{\xi} \Theta_o\,, \quad \int_{-\infty}^{\xi} \bar{\Theta}_o\,, \quad \int_{-\infty}^{\xi} \mathcal{J}_o\,, \quad \int_{-\infty}^{\xi} \bar{\mathcal{J}}_o \quad \in L^1_{loc}\big(\mathbb{R}_x \times \mathbb{R}^+_t \times \mathbb{R}_\xi\big)\,.$$

If we set

$$m = \partial_t \int_{-\infty}^{\xi} \Theta_o + \partial_x \int_{-\infty}^{\xi} \mathcal{J}_o\,, \quad \bar{m} = \partial_t \int_{-\infty}^{\xi} \bar{\Theta}_o + \partial_x \int_{-\infty}^{\xi} \bar{\mathcal{J}}_o\,,$$

then m, $\bar{m}$ satisfy (5.49). Note that (5.49) determines $\partial_\xi m$ and $\partial_\xi \bar{m}$ uniquely, and, in turn, m and $\bar{m}$ are uniquely determined in the class of bounded, positive measures.

Then, using Proposition 5.3 and (5.49), we express (5.2) in the form

$$\begin{aligned} &(m, \varphi \otimes p_{\xi\xi}) + (\bar{m}, \varphi \otimes \bar{p}_{\xi\xi}) \\ &\qquad = \int_{\mathbb{R}_x \times [0,T]} \int_{\mathbb{R}_\xi} \big(\Theta_o p_\xi + \bar{\Theta}_o \bar{p}_\xi\big)\varphi_t + \big(\mathcal{J}_o p_\xi + \bar{\mathcal{J}}_o \bar{p}_\xi\big)\varphi_x \, d\xi dx dt \\ &\qquad\quad + \int_{\mathbb{R}_x} \int_{\mathbb{R}_\xi} \big(\Theta_o(u_0(x), v_o(x), \xi) p_\xi + \bar{\Theta}_o(u_0(x), v_o(x), \xi) \bar{p}_\xi\big)\varphi(x,0)\, d\xi dx \ge 0 \end{aligned} \tag{5.51}$$

for $\varphi \in C^\infty_c(\mathbb{R}_x \times [0,T))$ with $\varphi \ge 0$ and $p, \bar{p} \in C^2(\mathbb{R}_\xi)$ and convex, where $(\cdot,\cdot)$ denotes the pairing between distributions and test functions. From here we conclude that m and $\bar{m}$ are positive measures.

We now turn to the proof of (3). Testing (5.51) against $p = \bar{p} = \frac{1}{2}\xi^2$ we see from (5.39) that

$$\begin{aligned} (m + \bar{m}, \varphi \otimes 1) = &\int_{\mathbb{R}_x \times [0,T]} \big[\frac{1}{2} v^2 + W(u)\big]\varphi_t - v(\sigma(u) - \sigma(0))\varphi_x \, dx dt \\ &+ \int_{\mathbb{R}_x} \big[\frac{1}{2} v_0^2(x) + W(u_0(x))\big]\varphi(x,0)\, dx \end{aligned}$$

To deduce (5.42) we choose a family of nonincreasing functions $\varphi_n(t,x) \geq 0$ which converges monotonically to 1 as n tends to infinity, and such that, for example, $\varphi_{n,x}$ vanishes in L^2 as in [LPT$_1$, Thm 1].

To prove (4), following the argument in the proof of (1), the growth rate of the test function in ξ is irrelevant. For instance, we may choose the test function $p(\xi) = \frac{1}{2}(\xi - M)_+^2$ with $M = \| v + |A(u)| \|_{L^\infty}$. From the support in ξ of the functions Θ_o, $\bar{\Theta}_o$, see Proposition 5.1 (iii), we deduce that $\int p''(\xi) m \, d\xi = 0$ and thus, we obtain $m = 0$ for $\xi \geq M$. The other bounds on the support of m, $\bar{m}$ follow in the same way. Finally, if (u,v) is a C^1 solution of (5.1) in some domain Ω, then it is reversible and satisfies (5.2) for any smooth entropy pair $\eta - q$. In turn, (5.51) holds as equality for all test functions p and $\bar{p}$. Since m and $\bar{m}$ are bounded measures we conclude that $m = \bar{m} = 0$ for $(x,t) \in \Omega$, $\xi \in \mathbb{R}$. □

Remark 5.8.

1. Theorem 5.6 extends the kinetic formulations for scalar conservation laws [LPT$_1$] and for the equations of isentropic gas dynamics [LPT$_2$]. As in the latter case (5.41) depends on the moments and does not provide a purely kinetic formalism. But contrary to gas dynamics, (5.41) form a system of two equations. This phenomenon was also pointed out for the $n \times n$ chromatography system which produces a family of n semi-kinetic equations (see [JPP]).

2. *Comparison with linear elasticity.* It is instructive to compare the form (5.11) of the kinetic functions for nonlinear stress-strain laws with that of linear elasticity: $\sigma(u) = a_o^2 u$, with $a_o > 0$ constant. For linear elasticity one computes (from Proposition 4.6) that $H = 1$, $Q = -a_o$, $\bar{H} = 1$ and $\bar{Q} = a_o$. The kinetic functions then become

$$\Theta_o = \mathbb{1}_{v+a_o u}(\xi)\,, \quad \mathcal{J}_o = -a_o \mathbb{1}_{v+a_o u}(\xi)\,, \qquad \bar{\Theta}_o = \mathbb{1}_{v-a_o u}(\xi)\,, \quad \mathcal{J}_o = a_o \mathbb{1}_{v-a_o u}(\xi)\,.$$

and thus the effects of each wave speed decouple. By contrast, for nonlinear laws, there is coupling through the factors κ_o and $\bar{\kappa}_o$, *cf.* (5.9) and (4.50)-(4.53).

3. *Invariant regions.* The classical property of invariant regions for the equations of elasticity under (c) can be obtained directly from the entropy formulation (5.2), (see Dafermos [Da$_1$], Serre [Se$_2$] for an independent proof). We outline their derivation from the kinetic formulation. Equation $(5.41)_1$, when tested against the test function $\phi(\xi - \xi_0)$ with ϕ convex, yields:

$$\partial_t \int_{\mathbb{R}} \Theta_o \partial_\xi \phi(\xi - \xi_0) d\xi + \partial_x \int_{\mathbb{R}} \mathcal{J}_o \partial_\xi \phi(\xi - \xi_0) d\xi = - < m, \partial_{\xi\xi} \phi(\xi - \xi_0) > \leq 0\,.$$

Using (5.27) that implies

$$\int_{\mathbb{R}} < \mathcal{X}(u(x,t), v(x,t) - \xi), \phi(\xi - \xi_o) > dx \leq \int_{\mathbb{R}} < \mathcal{X}(u_0(x), v_0(x) - \xi), \phi(\xi - \xi_0) > dx\,, \tag{5.52}$$

where the brackets denote the action of $\mathcal{X}(u, v - \cdot)$ on $\phi(\cdot - \xi_0)$ (and should be understood as defined through integration by parts in ξ).

Choose $\xi_0 = \sup_x\{v_0 + |A(u_0)|\}$ and $\phi(\xi - \xi_o) = [(\xi - \xi_0)_+]^2$. Since $\mathcal{X}$ is a positive measure with support as in (5.19), the right hand side in (5.52) equals zero and the resulting inequality gives

$$v(x,t) + |A(u(x,t))| \leq \sup_x\{v_0 + |A(u_0)|\}\,. \tag{5.53}$$

The choices $\bar{\xi}_0 = \inf_x\{v_0 - |A(u_0)|\}$ and $\phi(\xi - \xi_0) = [(\xi - \bar{\xi}_0)_-]^2$ provide the complementary inequality

$$v(x,t) - |A(u(x,t))| \geq \inf_x\{v_0 - |A(u_0)|\}\,. \tag{5.54}$$

4. L^p *solutions.* We finish with some remarks concerning the kinetic formulation for L^p solutions, always under hypothesis (c). The kinetic functions $\Theta_o - \mathcal{J}_o$, $\bar{\Theta}_o - \bar{\mathcal{J}}_o$ are in $L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x \times \mathbb{R}_\xi)$ under quite weak hypotheses: (u, v) a measurable function with $a(u) \in L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x)$. Nevertheless, (5.41) becomes meaningful when tested against convex test functions, and the growth rate of these test functions (as $|\xi| \to \infty$) reflects on the growth of the resulting entropy pairs (see Lemma 5.7).

The proof of Theorem 5.6 will work for L^p solutions with minor modifications. Instead of formulating a precise statement, we allude to the points that require care for L^p solutions. For part (i), the extension from (5.47) to (5.48) requires to test with test functions of linear growth and (by (5.46)) requires integrability of solutions. It is seen clearly from the proof that the growth rate of allowable test functions is directly connected with the growth of entropy entropy-flux pair in (u, v) and ultimately to the integrability of the solution. For part (ii), in order to define m and $\bar{m}$ it is at least needed that $v, A(u), a(u)v, a(u)A(u) \in L^1_{loc}(\mathbb{R}_x \times \mathbb{R}^+_t)$, see (5.50).

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References

[Ba] Ball J.M., A version of the fundamental theorem for Young measures, In *PDEs and Continuum Models of Phase Transitions* , M. Rascle, D. Serre and M. Slemrod, eds., Lecture Notes in Physics, Vol. 344, Springer, New York, 1988, pp. 207-215.

[Br] Brenier Y., Résolution d' equations d' évolution quasilinéaires en dimension N d'espace à l'aide déquations linéaires en dimension $N+1$, *J. Differential Equations* **50** (1983), 375-390.

[CL] Chen G.-Q. and LeFloch Ph., Compressible Euler equations with general pressure laws, (preprint).

[Da$_1$] Dafermos C.M., Estimates for conservation laws with little viscosity, *SIAM J. Math. Anal.* **18** (1987), 409-421.

[Da_2] Dafermos C.M., Generalized characteristics in hyperbolic systems of conservation laws, *Arch. Rational Mech. Analysis* **107** (1989), 127-155.

[DM] Dellacherie C. and Meyer P.-A., *Probabilités et Potentiel*, Hermann, 1975.

[DCL] Ding X., Chen G.-Q. and Luo P., Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (I) and (II), *Acta Mathematica Scientia* **5** (1985), 415-432; and **7** (1987), 433-472.

[Dp_1] DiPerna R., Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Analysis* **82** (1983), 75-100.

[Dp_2] DiPerna R., Convergence of the viscosity method for isentropic gas dynamics *Comm. Math. Phys.* **91** (1983), 1-30.

[DS] Dunford N. and Schwartz J.T., *Linear Operators*, Part I, Interscience, 1967.

[JPP] James F., Peng Y.-J. and Perthame, B., Kinetic formulation for chromatography and some other hyperbolic systems. *J. Math. Pures et Appl.* **74** (1995), 367-385.

[La] Lax P.D., Shock waves and entropy, in: "Contributions to Nonlinear Functional Analysis." E.H. Zarantonello, ed. New York: Academic Press, 1971, pp. 603-634.

[Lin] Lin P., Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics. *Trans. Amer. Math. Soc.* **329** (1992), 377-413.

[LPT_1] Lions P.L., Perthame B. and Tadmor E., A kinetic formulation of scalar multidimensional conservation laws, *J. AMS* **7** (1994), 169-191.

[LPT_2] Lions P.L., Perthame B. and Tadmor E., Kinetic formulation of the isentropic gas dynamics and p-systems, *Comm. Math. Physics* **163** (1994), 415-431.

[LPS] Lions P.L., Perthame B. and Souganidis P.E., Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates, *Comm. Pure Appl. Math.* **49** (1996), 599-638.

[NOV] Nouri A., Omrane A. and Vila J.P., Boundary conditions for scalar conservation laws from a kinetic point of view, to appear in *J. Stat. Phys.*.

[PT] Perthame B. and Tadmor E., A kinetic equation with kinetic entropy functions for scalar conservation laws, *Comm. Math. Phys.* **136** (1991), 501-517.

[S] Schonbek M.E., Convergence of solutions to nonlinear dispersive equations, *Comm. Partial Differential Equations* **7** (1982), 959-1000.

[Se_1] Serre D., La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d' espace. *J. Maths. Pures et Appl.* **65** (1986), 423-468.

[Se_2] Serre D., Domaines invariants pour les systèmes hyperboliques de lois de conservation. *J. Differential Equations* **69** (1987), 46-62.

[Se_3] Serre D., *Systèmes de lois de conservation I, II.* Fondations, Diderot, Paris, 1996.

[Sh] Shearer J., Global existence and compactness in L^p for the quasi-linear wave equation. *Comm. Partial Differential Equations* **19** (1994), 1829-1877.

[Ta] Tartar L., Compensated compactness and applications to partial differential equations, In *Nonlinear Analysis and Mechanics, Heriot Watt Symposium, Vol. IV*, R.J. Knops, ed., Pitman Research Notes in Math., New York, 1979, pp. 136–192.

[Va] Vasseur A., Time regularity for the system of isentropic gas dynamics with $\gamma = 3$. *Comm. Partial Differential Equations* **24** (1999), 1987-1997.