STRESS RELAXATION MODELS WITH POLYCONVEX ENTROPY IN LAGRANGEAN AND EULERIAN COORDINATES

ATHANASIOS E. TZAVARAS

To Marshall Slemrod on the occasion of his 70th birthday with friendship and admiration

Abstract. The embedding of the equations of polyconvex elastodynamics to an augmented symmetric hyperbolic system provides in conjunction with the relative entropy method a robust stability framework for approximate solutions [18]. We devise here a model of stress relaxation motivated by the format of the enlargement process which formally approximates the equations of polyconvex elastodynamics. The model is endowed with an entropy function which is not convex but rather of polyconvex type. Using the relative entropy we prove a stability estimate and convergence of the stress relaxation model to polyconvex elastodynamics in the smooth regime. As an application, we show that models of pressure relaxation for real gases in Eulerian coordinates fit into the proposed framework.

1. Introduction

The mechanical motion of a continuous medium with nonlinear elastic response is described by the system of partial differential equations

$$\frac{\partial^2 y}{\partial t^2} = \nabla \cdot T(\nabla y) \tag{1.1}$$

where $y : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ describes the motion, T is the Piola–Kirchhoff stress tensor, $v = \partial_t y$ is the velocity and $F = \nabla y$ the deformation gradient. Motivated by the requirements imposed on the theory of thermoelasticity from consistency with the Clausius-Duhem inequality of thermodynamics, one often imposes the assumption of hyperelasticity, i.e. that T is expressed as a gradient $T(F) = \frac{\partial W}{\partial F}(F)$ of the stored energy function $W : \mathrm{Mat}^{3\times 3} \to [0, \infty)$. The principle of material frame indifference dictates that W remains invariant under rotations

$$W(OF) = W(F) \quad \text{for all orthogonal matrices } O \in O(3).$$

Convexity of the stored energy W is too restrictive and even incompatible with certain physical requirements: First, it conflicts with frame indifference in conjunction with the requirement that the energy increase without bound as $\det F \to 0^+$. Second, convexity of the energy together with the axiom of frame indifference impose restrictions on the induced Cauchy stresses that rule out certain naturally occurring states of stress (*e.g.* [8, Sec 8], [6, Sec 4.8]). As a result, it has been replaced in the theory of elastostatics by various weaker notions such as quasi-convexity, rank-1 convexity or polyconvexity, see [1] or [2] for a recent survey. Here, we adopt the assumption of polyconvexity which postulates that

$$W(F) = g(F, \mathrm{cof}\, F, \det F)\,,$$

2000 *Mathematics Subject Classification.* 35L65, 36L75, 74B20, 82C40.
Key words and phrases. polyconvex elasticity, relaxation limits, pressure relaxation in gases.

where g is a strictly convex function of $\Phi(F) = (F, \operatorname{cof} F, \det F)$, and encompasses various interesting models (*e.g.* [6]).

The system (1.1) may be recast as a system of conservation laws, for the velocity $v = \partial_t y$ and the deformation gradient $F = \nabla y$, in the form

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F)\,, \end{aligned} \tag{1.2}$$

$i, \alpha = 1, \dots, 3$. The equivalence holds for solutions (v, F) with $F = \nabla y$, i.e. subject to the set of differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,. \tag{1.3}$$

Equation (1.3) is an involution: if it holds initially it is propagated by $(1.2)_1$ to hold for all times. The system (1.2) is endowed with an additional conservation law

$$\partial_t(\frac{1}{2}v^2 + W(F)) - \partial_\alpha(v_i T_{i\alpha}(F)) = 0 \tag{1.4}$$

manifesting the conservation of mechanical energy. When W is convex the "entropy" $E = \frac{1}{2}v^2 + W(F)$ is a convex function. Convexity of the entropy is known to provide a stabilizing mechanism for thermomechanical processes, and entropy inequalities for convex entropies have been employed in the theory of hyperbolic conservation laws as an admissibility criterion for weak solutions [19] and provide powerful stability frameworks for approximations of classical solutions [11], [16]. Such stability is attained via the relative entropy method and applies in particular to viscosity or even relaxation approximations of the system (1.2), [18], [13, Ch V].

By contrast, when W is not convex the entropy $E = \frac{1}{2}v^2 + W(F)$ is also non-convex, what induces an array of questions regarding the stability of the model within its various approximating theories. One should distinguish between models where one tries to model inherently unstable phenomena (like for example phase transitions) and models where one expects stable response but where the invariance under rotations imposes degeneracies (like the problem of elasticity). Our objective is to contribute to a program [20, 14, 18] of understanding such issues and to suggest remedies especially as it pertains to the stable approximation of elastodynamics by stress relaxation theories.

Relaxation approximations encompass many physical models and have proved useful in designing efficient algorithms for systems of conservation laws (*e.g.* [10, 3, 5]) while convexity of the entropy is known to provide a stabilizing effect for general relaxation approximations (*e.g* [7], [25]). A natural relaxation approximation of (1.2) is given by the stress relaxation theory

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t(S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\varepsilon}(S_{i\alpha} - T_{i\alpha}(F)). \end{aligned} \tag{1.5}$$

This model may be visualized within the framework of viscoelasticity with memory

$$S = f(F) + \int_{-\infty}^{t} \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(t-\tau)} h(F(\cdot, \tau))\, d\tau$$

with the equilibrium stress $T(F)$ decomposed into an elastic and viscoelastic contribution, $T(F) = f(F) + h(F)$, where $f = \frac{\partial W_I}{\partial F}$ and $T = \frac{\partial W}{\partial F}$, and a kernel exhibiting a single relaxation time ε. It belongs to the class of thermomechanical theories with internal variables which have been extensively studied in the mechanics literature *e.g.* [9, 17, 23, 24].

The approximation (1.5) is consistent with the second law of thermodynamics, provided the potential of the instantaneous elastic response W_I dominates the potential of the equilibrium response W. When W is convex the relaxation theory has a convex entropy and a relative entropy calculation indicates that (1.5) stably approximates (1.2) [18]. On the other hand, for polyconvex W, the consistency with thermodynamics is still attained but the entropy of the relaxation system loses convexity and the stability of the approximating system is questionable. Convexity of the entropy is a dictum of stability for relaxation approximations; at the same time it is not a consequence of thermodynamical consistency of relaxation theories with the Clausius-Duhem inequality [23, 18]. As convexity is largely incompatible with material frame indifference, the effect of adopting weaker notions of convexity on the stability of thermomechanical processes needs to be further understood.

Our objective is to propose a stable relaxation approximation scheme for the equations of polyconvex elasticity. We will be guided by the embedding of polyconvex elasticity to an augmented strictly hyperbolic system: Due to nonlinear transport identities of the null-Lagrangians, the system (1.2) with polyconvex stored energy can be embedded into an augmented symmetric hyperbolic system

$$
\begin{aligned}
\partial_t v_i &= \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\
\partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right)
\end{aligned}
\tag{1.6}
$$

and be visualized as constrained evolution thereof (see [20, 14] and section 2 for an outline). The augmented system admits the convex entropy $\eta = \frac{1}{2}|v|^2 + g(\Xi)$ and is symmetrizable. The idea of symmetrizable extensions of (1.2) has important implications on the equations of polyconvex elasticity, providing stability frameworks between entropy weak and smooth solutions [18], [13, Ch V] or even between entropic measure-valued and smooth solutions [15]. The idea of enlarging the number of variables and extending to symmetrizable hyperbolic systems has been fruitful in other contexts like for nonlinear models of electromagnetism [4, 21] or for the isometric embedding problem in geometry [22].

In the sequel, we consider the stress relaxation system

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \left(T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\
\partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\
\partial_t \left(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Phi(F)) \right) &= -\frac{1}{\varepsilon} \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \right) \\
\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} &= 0\,.
\end{aligned}
\tag{1.7}
$$

The format of (1.7) is motivated by an attempt to transfer the geometric structure of the limit to the approximating relaxation system. Note that (1.7) formally approximates as $\varepsilon \to 0$ the equations of polyconvex elastodynamics and retains the property of embedding into to an augmented relaxation system (see (3.6)) with the latter endowed with an entropy dissipation inequality for a convex entropy. The reduced entropy inherited by (1.7) is of the form

$$\mathcal{E} = \frac{1}{2}|v|^2 + \Psi(\Phi(F), \tau)$$

with $\Psi(\Xi, \tau)$ a convex function, $\Phi(F) = (F, \operatorname{cof} F, \det F)$, and thus $\mathcal{E}$ is of polyconvex type. We prove using a relative entropy computation and the null-Lagrangian structure that this theory approximates in a stable way smooth solutions of (1.2) with polyconvex stored energy. Our analysis indicates that it is possible to stabilize a relaxation model via a globally defined, polyconvex entropy.

The system (1.7) appears unconventional as it mixes geometric and mechanical properties. Nevertheless, it contains a very interesting example. When the equations of isentropic gas dynamics in Eulerian coordinates are adapted to this model, and after performing the proper transformations from Eulerian to Lagrangean coordinates, one achieves a model of relaxation of pressures (see (5.19)) with an instantaneous and an equilibrium pressure response. The latter is endowed with a globally defined, dissipative, convex entropy. Models of pressure relaxation have been considered before in [9, 23]. The novelty of the present one is the existence of a global, convex entropy. This is in a similar spirit (but a different model) as the model of internal energy relaxation for gas dynamics pursued in [10].

The article is organized as follows. In Section 2 we present the embedding of (1.2) into the augmented system (1.6) and define the relative entropy. In section 3 we state the augmented relaxation system (3.6), show that it is endowed with a convex entropy, and exhibit the inherited relative entropy calculation (3.24) for the system (1.7). This culminates into the stability and convergence Theorem 4.1 between solutions of the relaxation model (1.7) and the polyconvex elastodynamics system (1.2). As an application of the theory, in section 5, we develop an example of pressure relaxation that converges to the equations of isentropic gas dynamics in Eulerian coordinates and is endowed with a convex entropy function.

The results of sections 3 and 4 are taken from an earlier unpublished version of this manuscript [26].

2. The symmetrizable extension of polyconvex elastodynamics

The system of elastodynamics (1.1) is expressed in the form of a system of conservation laws (1.2), (1.3). As already noted, the equivalence of the two formulations holds for functions F that are gradients, but as the relation $F = \nabla y$ propagates from the initial data, relation (1.3) is viewed as a constraint on the initial data and is usually omitted. We work under the framework of polyconvex hyperelasticity: the Piola-Kirchoff stress is derived from a potential $T(F) = \frac{\partial W(F)}{\partial F}$ and the stored energy $W : \mathrm{Mat}^{3\times 3} \to [0, \infty)$ factorizes as a function of the minors of F,

$$W(F) = \big(g \circ \Phi\big)(F)\,, \quad \text{where } \Phi(F) = (F, \operatorname{cof} F, \det F)\,, \tag{2.1}$$

with $g : \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ convex. The cofactor matrix $\operatorname{cof} F$ and the determinant $\det F$ are

$$\begin{aligned}
(\operatorname{cof} F)_{i\alpha} &= \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}F_{k\gamma}\,, \\
\det F &= \frac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{i\alpha}F_{j\beta}F_{k\gamma} = \frac{1}{3}(\operatorname{cof} F)_{i\alpha}F_{i\alpha}\,.
\end{aligned}$$

We review a symmetrizable extension of polyconvex elastodynamics [14], based on certain kinematic identities on $\det F$ and $\operatorname{cof} F$ from [20]. The components of $\Phi(F)$ are null Lagrangians and satisfy the identities

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla y)\right) \equiv 0$$

for any smooth map $y(x,t)$. Equivalently, this is expressed as

$$\partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) = 0\,, \quad \forall F \text{ with } \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,. \tag{2.2}$$

The kinematic compatibility equation $(1.2)_1$ implies

$$\partial_t \Phi^A(F) - \partial_\alpha\left(v_i \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) = 0\,. \tag{2.3}$$

Strictly speaking (2.3) do not form what is called in the theory of conservation laws entropy - entropy flux pairs as they hold only for F that are gradients, i.e. $\forall F$ with $\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0$.

This motivates to embed (1.2), (2.3) into the system of conservation laws

$$\begin{aligned} \partial_t v_i &= \partial_\alpha\left(\frac{\partial g}{\partial \Xi^A}(\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \\ \partial_t \Xi^A &= \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i\right)\,. \end{aligned} \tag{2.4}$$

Note that $\Xi = (F, Z, w)$ takes values in $\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \simeq \mathbb{R}^{19}$ and is treated as a new dependent variable. The extension has the following properties:

(i) If $F(\cdot, 0)$ is a gradient then $F(\cdot, t)$ remains a gradient $\forall t$.

(ii) If $\Xi(\cdot, 0) = \Phi(F(\cdot, 0))$ with $F(\cdot, 0) = \nabla y_0$, then $\Xi(\cdot, t) = \Phi(F(\cdot, t))$ where $F(\cdot, t) = \nabla y(\cdot, t)$. In other words, the system of elastodynamics can be visualized as constrained evolution of (2.4).

(iii) The enlarged system admits a strictly convex entropy

$$\eta(v, \Xi) = \frac{1}{2}|v|^2 + g(\Xi)$$

and is thus symmetrizable (along solutions that are gradients).

(iv) The system is endowed with a relative entropy calculation, detailed below.

Property (iii) is based on the null-Lagrangian structure and η is not an entropy in the usual sense of the theory of conservation laws. Rather, the identity

$$\partial_t\left[\frac{1}{2}|v|^2 + g(\Xi)\right] - \partial_\alpha\left[\sum_{i,A} v_i \frac{\partial g(\Xi)}{\partial \Xi^A}\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\right] = 0 \tag{2.5}$$

holds for F's that are gradients.

Property (iv) pertains to the following **relative energy** calculation [18], [13, Ch V]. Let y be an entropic weak solution satisfying the weak form of (1.2), (1.3) and the weak form of the entropy inequality

$$\partial_t \left[\frac{1}{2}|v|^2 + g(\Phi(F)) \right] - \partial_\alpha \left[\sum_{i,A} v_i \frac{\partial g(\Phi(F))}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right] \le 0 \quad \text{in } \mathcal{D}'.$$

Then provided $F = \nabla y$ enjoys sufficient integrability properties, F also satisfies the weak form of (2.3). As a result (v, Ξ) with $\Xi = \Phi(F)$ is a weak solution of (2.4) which is entropic in the sense that

$$\partial_t \left[\frac{1}{2}|v|^2 + g(\Xi) \right] - \partial_\alpha \left[\sum_{i,A} v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right] \le 0 \quad \text{in } \mathcal{D}'.$$

Let $\widehat{y}$ be a smooth solution of (1.1). Then $(\widehat{v}, \widehat{F})$ satisfy (1.2), (1.3) and the augmented function $(\widehat{v}, \widehat{\Xi})$ with $\widehat{\Xi} = \Phi(\widehat{F})$ satisfies the energy conservation (2.5). Then, the two solutions $(v, \Phi(F))$ and $(\widehat{v}, \Phi(\widehat{F}))$ can be compared via the relative energy formula

$$\partial_t \Big(\eta(v, \Phi(F) \mid \widehat{v}, \Phi(\widehat{F})) \Big) - \nabla \cdot \Big(q(v, \Phi(F) \mid \widehat{v}, \Phi(\widehat{F})) \Big) \le Q\,, \tag{2.6}$$

where

$$\begin{aligned} \eta(v, \Xi \mid \widehat{v}, \widehat{\Xi}) &:= \frac{1}{2}|v - \widehat{v}|^2 + g(\Xi) - g(\widehat{\Xi}) - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A}(\Xi^A - \widehat{\Xi}^A)\,, \\ q^\alpha(v, \Xi \mid \widehat{v}, \widehat{\Xi}) &:= \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right)(v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}\,, \quad \alpha = 1, 2, 3\,, \end{aligned}$$

and Q is a quadratic error term of the form

$$\begin{aligned} Q := &\Big[\frac{\partial^2 g}{\partial \Xi^A \partial \Xi^B}(\Phi(\widehat{F})) \Big] \partial_\alpha(\Phi^B(\widehat{F})) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right)(v_i - \widehat{v}_i) \\ &+ (\partial_\alpha \widehat{v}_i) \left(\frac{\partial g(\Phi(F))}{\partial \Xi^A} - \frac{\partial g(\Phi(\widehat{F}))}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\ &+ (\partial_\alpha \widehat{v}_i) \Big(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B}(\Xi^B - \widehat{\Xi}^B) \Big) \Big|_{\Xi = \Phi(F),\, \widehat{\Xi} = \Phi(\widehat{F})} \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}}\,. \end{aligned} \tag{2.7}$$

Details of the lengthy computation can be found in [18] and use in a substantial way the null-Lagrangian identity (2.2). There is also available an analogous formula for comparing entropic (or dissipative) measure-valued solutions to smooth solutions of (1.2), see [15].

3. A relaxation model for polyconvex elastodynamics

We next consider the stress relaxation model

$$\begin{aligned} \partial_t v_i - \partial_\alpha \left(T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\ \partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\ \partial_t \left(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Phi(F)) \right) &= -\frac{1}{\varepsilon} \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \right) \\ \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} &= 0\,, \end{aligned} \tag{3.1}$$

and wish to compare it to the equations of elastodynamics

$$\begin{aligned} \partial_t v_i - \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\ \partial_t F_{i\alpha} - \partial_\alpha v_i &= 0\,. \end{aligned} \tag{3.2}$$

The stress in the model (3.2) satisfies

$$S_\infty = \frac{\partial}{\partial F} \sigma_E(\Phi(F))$$

and thus, when σ_E is convex, the model (3.2) corresponds to polyconvex elasticity.

The model (3.1) corresponds to a stress relaxation theory where the stress is decomposed into an instantaneous and a viscoelastic part

$$S = T^A \frac{\partial \Phi^A}{\partial F} = \frac{\partial(\sigma_I \circ \Phi)}{\partial F} + \tau^A \frac{\partial \Phi^A}{\partial F} \tag{3.3}$$

and where the instantaneous elasticity is derived from a polyconvex potential $\sigma_I(\Phi(F))$ while the viscoelastic part is determined by internal variables τ^A evolving according to the model

$$\partial_t \tau^A = -\frac{1}{\varepsilon}\Big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi^A}(\Phi(F))\Big)\,. \tag{3.4}$$

Note that when expressed in terms of the motion y the model (3.1) takes the form

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \nabla \cdot \Big(\frac{\partial(\sigma_I \circ \Phi)}{\partial F}(\nabla y) + \tau^A \frac{\partial \Phi^A}{\partial F}(\nabla y) \Big) \\ \frac{\partial \tau^A}{\partial t} &= -\frac{1}{\varepsilon}\Big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi^A}(\Phi(\nabla y))\Big) \end{aligned} \tag{3.5}$$

Of course it may recast in the form of a theory with memory by integrating (3.4). We will see that the model (3.1) has very interesting structural properties.

3.1. **The augmented relaxation system.** The format of the stress relaxation model (3.1) is motivated (and was guided) by the enlargement structure of the polyconvex elastodynamics system (3.2) described in section 2.

Indeed, (3.1) can be embedded into the augmented relaxation system

$$\begin{aligned} \partial_t v_i - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} T^A \Big) &= 0 \\ \partial_t \Xi^A - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \Big) &= 0 \\ \partial_t \Big(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Xi) \Big) &= -\frac{1}{\varepsilon}\Big(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \Big) \end{aligned} \tag{3.6}$$

The stress function in the model (3.6) reads:

$$S_{i\alpha} = T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}}\,. \tag{3.7}$$

Note that as $\varepsilon \to 0$ the stress $S_{i\alpha}$ formally approximates the limiting stress

$$S_{i\alpha,\infty} = T^A(\Xi)\Big|_{eq} \frac{\partial \Phi^A}{\partial F_{i\alpha}} = \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}$$

and thus (3.6) will approximate the extended elastodynamics system

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) &= 0 \\
\partial_t \Xi^A - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \Big) &= 0
\end{aligned}
\tag{3.8}
$$

Observe also that solutions of (3.1) satisfy the kinematic constraints (2.3) and thus, for a polyconvex stored energy, the relaxation system (3.1) enjoys the same relation with the system (3.6) as the equations of polyconvex elastodynamics (1.2) have with the system (2.4).

Next, we develop the Chapman-Enskog expansion for the relaxation limit from (3.6) to (3.8). Introduce the expansion for the internal variable T^A

$$
T^{A,\varepsilon} = T_0^A + \varepsilon T_1^A + O(\varepsilon^2)
$$

and, accordingly,

$$
S_{i\alpha}^\varepsilon = T_0^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} + \varepsilon T_1^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} + O(\varepsilon^2)
$$

to (3.6) in order to obtain

$$
\begin{aligned}
T_0^A &= \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \\
\partial_t \Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_I}{\partial \Xi^A}(\Xi) \Big) &= -T_1^A + O(\varepsilon)
\end{aligned}
$$

The effective momentum equation becomes

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \Big(T_0^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) &= \varepsilon \partial_\alpha \Big(T_1^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) + O(\varepsilon^2) \\
&= \varepsilon \partial_\alpha (D_{i\alpha}^{j\beta} \partial_\beta v_j) + O(\varepsilon^2)
\end{aligned}
$$

where

$$
D_{i\alpha}^{j\beta} := \frac{\partial^2 (\sigma_I - \sigma_E)}{\partial \Xi^A \partial \Xi^B} \frac{\partial \Phi^A}{\partial F_{i\alpha}} \frac{\partial \Phi^B}{\partial F_{j\beta}}
\tag{3.9}
$$

In summary, the Chapman-Enskog expansion shows that as $\varepsilon \to 0$ the relaxation process is approximated by the hyperbolic-parabolic system

$$
\begin{aligned}
\partial_t \Xi^A - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \Big) &= 0 \\
\partial_t v_i - \partial_\alpha \Big(T_0^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) &= \varepsilon \partial_\alpha (D_{i\alpha}^{j\beta} \partial_\beta v_j)
\end{aligned}
$$

Note that for $\Sigma := \sigma_I - \sigma_E$ convex the diffusivity tensor D satisfies the ellipticity condition $D_{i\alpha}^{j\beta} M_{i\alpha} M_{j\beta} \geq 0$, $\forall M \in \mathbb{R}^{3\times 3}$. The latter is stronger than the Legendre-Hadamard condition, and is achieved when both the instantaneous potential $\sigma_I \circ \Phi$ and the equilibrium potential $\sigma_E \circ \Phi$ are polyconvex.

3.2. **Entropy of the augmented relaxation system.** We next construct an entropy for the augmented relaxation system: If a function $\Psi(\Xi,\tau)$ can be constructed defined $\forall\,(\Xi,\tau)$ and satisfying

$$\begin{aligned}\frac{\partial\Psi}{\partial\Xi^A}(\Xi,\tau) = T^A = \frac{\partial\sigma_I(\Xi)}{\partial\Xi^A} + \tau^A \\ \frac{\partial\Psi}{\partial\tau^A}\Big(\tau^A - \frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi^A}\Big) \geq 0 \qquad \forall\,(\Xi,\tau)\,,\end{aligned} \tag{3.10}$$

then the relaxation system is endowed with an H-theorem

$$\partial_t\Big(\frac{1}{2}|v|^2 + \Psi(\Xi,\tau)\Big) - \partial_\alpha\big(v_iS_{i\alpha}\big) + \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial\tau^A}\big(\tau^A - \frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi^A}\big) = 0\,. \tag{3.11}$$

This entropy identity is based on the null-Lagrangian property (2.2) and follows, using (3.6), (2.2) and (3.10), by the computation

$$\begin{aligned}\partial_t\Big(\frac{1}{2}|v|^2 + \Psi(\Xi,\tau)\Big) &= v_i\partial_t v_i + \frac{\partial\Psi}{\partial\Xi^A}\partial_t\Xi^A + \frac{\partial\Psi}{\partial\tau^A}\partial_t\tau^A \\ &= v_i\partial_\alpha S_{i\alpha} + \frac{\partial\Psi}{\partial\Xi^A}\partial_\alpha\big(\frac{\partial\Phi^A}{\partial F_{i\alpha}}v_i\big) + \frac{\partial\Psi}{\partial\tau^A}\partial_t\tau^A \\ &= v_i\partial_\alpha S_{i\alpha} + \frac{\partial\Psi}{\partial\Xi^A}\frac{\partial\Phi^A}{\partial F_{i\alpha}}\partial_\alpha v_i + \frac{\partial\Psi}{\partial\tau^A}\partial_t\tau^A \\ &= \partial_\alpha(v_iS_{i\alpha}) - \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial\tau^A}\big(\tau^A - \frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi^A}\big)\end{aligned}$$

Our next objective is to examine the solvability of (3.10) and study the convexity of the entropy. Integrating $(3.10)_1$, we see that

$$\Psi(\Xi,\tau) = \sigma_I(\Xi) + \Xi\cdot\tau + G(\tau) \tag{3.12}$$

where the integrating factor $G(\tau)$ has to be selected so that it satisfies the inequality

$$(\Xi + \nabla_\tau G)\cdot(\tau + \nabla_\Xi\Sigma) \geq 0 \quad \forall\,(\Xi,\tau) \tag{3.13}$$

where $\Sigma = \sigma_I - \sigma_E$. Regarding the solvability of (3.13), we show

Lemma 3.1. *The functions $G(\tau)$ and $\Sigma(\Xi)$ in $C^2(\mathbb{R}^m)$ satisfy (3.13) if and only if*

$$\begin{cases}\Xi + \nabla_\tau G = 0 \quad \textit{iff} \quad \tau + \nabla_\Xi\Sigma = 0 \\ G \textit{ is convex} \\ \Sigma \textit{ is convex}\end{cases} \tag{3.14}$$

Equation $(3.14)_1$ indicates that $G(\tau)$ and $\Sigma(\Xi)$ are connected through the Legendre transformation.

Proof. We first show that (3.13) implies (3.14). Fix Ξ^0, τ^0 such that $\Xi^0 + \nabla_\tau G(\tau^0) = 0$. Consider a fixed direction e^A and the increment along this direction $\Xi = \Xi^0 + te^A$. Then (3.13) implies that $e^A\cdot(\tau^0 + \nabla_\Xi\Sigma(\Xi^0)) = 0$ for every direction e^A and thus $\tau^0 + \nabla_\Xi\Sigma(\Xi^0) = 0$. Similarly, if Ξ^0, τ^0 are such that $\tau^0 + \nabla_\Xi\Sigma(\Xi^0) = 0$ then also $\Xi^0 + \nabla_\tau G(\tau^0) = 0$. This proves the first statement in (3.14).

Fix now Ξ_1, Ξ_2 and let $\tau_2 = -\nabla_\Xi\Sigma(\Xi_2)$. Then $\Xi_2 = -\nabla_\tau G(\tau_2)$, (3.13) is rewritten as

$$(\Xi_1 - \Xi_2)\cdot\big(\nabla_\Xi\Sigma(\Xi_1) - \nabla_\Xi\Sigma(\Xi_2)\big) \geq 0 \tag{3.15}$$

and Σ is convex. A similar argument shows that G is convex.

The converse is proved by re-expressing the convexity inequality (3.15) in the form (3.13) by using the first statement in the right of (3.14). $\square$

Lemma 3.1 indicates that the solvability of (3.10) is equivalent to the convexity of $\Sigma := \sigma_I - \sigma_E$. To complete the details of the construction of Ψ, we assume for simplicity that

$$\nabla^2_\Xi \Sigma > 0 \quad \text{and} \quad \nabla_\Xi \Sigma : \mathbb{R}^D \to \mathbb{R}^D \text{ is onto,} \tag{h$_0$}$$

with $D = 19$ for $d = 3$ and $D = 5$ for $d = 2$. Define the inverse map $(\nabla_\Xi \Sigma)^{-1} : \mathbb{R}^D \to \mathbb{R}^D$, and let $h(\tau) = -(\nabla_\Xi \Sigma)^{-1}(-\tau)$. Then $\nabla_\tau h$ is symmetric and the differential system $\nabla_\tau G = h$ is solvable. Its solution G is a convex function and satisfies

$$\begin{aligned} \nabla_\tau G(\tau) &= -\left(\nabla_\Xi \Sigma\right)^{-1}(-\tau) \\ \nabla^2_\tau G(\tau) &= \left[\nabla^2_\Xi \Sigma(-\nabla_\tau G)\right]^{-1} \end{aligned} \tag{3.16}$$

Ψ is defined by (3.12) with G as above. Observe that, by (3.10) and (3.14),

$$\begin{aligned} \frac{\partial \Psi}{\partial \Xi^A}(\Xi, -\nabla_\Xi \Sigma) &= \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \\ \frac{\partial \Psi}{\partial \tau^A}(\Xi, -\nabla_\Xi \Sigma) &= \Xi^A + \frac{\partial G}{\partial \tau^A}\Big|_{\tau^A = -\frac{\partial(\sigma_I - \sigma_E)}{\partial \Xi^A}} = 0 \end{aligned} \tag{3.17}$$

and, by selecting a normalization constant,

$$\Psi(\Xi, -\nabla_\Xi \Sigma) = \sigma_E(\Xi) \tag{3.18}$$

We next consider the convexity of $\Psi(\Xi, \tau)$ determined by the matrix

$$\nabla^2_{(\Xi,\tau)} \Psi = \begin{bmatrix} \nabla^2_\Xi \sigma_I & \mathbb{I} \\ \mathbb{I} & \nabla^2_\tau G \end{bmatrix}$$

Lemma 3.2. *Let $\Sigma = \sigma_I - \sigma_E$ satisfy* (h$_0$) *and assume that σ_I, Σ satisfy for $\gamma_I > \gamma_v > 0$*

$$\nabla^2_\Xi \sigma_I \geq \gamma_I \mathbb{I}_m > \gamma_v \mathbb{I}_m \geq \nabla^2_\Xi \Sigma > 0 \tag{h$_1$}$$

Then for some $\delta > 0$ we have

$$\nabla^2_{(\Xi,\tau)} \Psi \geq \delta \, \mathbb{I}_{(\Xi,\tau)}$$

Proof. Using (h$_1$) and $(3.16)_2$ we estimate the Hessian of Ψ as follows

$$\begin{aligned} \left(\nabla^2_{(\Xi,\tau)} \Psi\right) \, (\Xi, \tau) \cdot (\Xi, \tau) &= (\nabla^2_\Xi \sigma_I) \Xi \cdot \Xi + 2\Xi \cdot \tau + \left(\nabla^2_\Xi \Sigma\right)^{-1} \tau \cdot \tau \\ &\geq \gamma_I |\Xi|^2 + 2\Xi \cdot \tau + \frac{1}{\gamma_v} |\tau|^2 \\ &\geq (\gamma_I - \delta)|\Xi|^2 + \Big(\frac{1}{\gamma_v} - \frac{1}{\delta}\Big)|\tau|^2 \,. \end{aligned}$$

The coefficients can be made positive definite by selecting $\gamma_I > \delta > \gamma_v$. $\square$

Remark 3.3. Hypothesis (h$_1$) implies that σ_E must be convex, which dictates that the limiting equations arise from a polyconvex energy.

3.3. **Relative entropy for the augmented system.** Next we compare a solution (v, Ξ, τ) of the system (3.6) with a solution $(\widehat{v}, \widehat{\Xi})$ of the extended elastodynamics system (3.8), using a relative entropy calculation in the spirit of [18, 25].

The relative entropy is defined by taking the Taylor polynomial of a nonequilibrium relative to a Maxwellian solution

$$\begin{aligned}\mathcal{E}_r := \frac{1}{2}|v-\hat{v}|^2 + \Psi(\Xi,\tau) - \Psi\Big(\widehat{\Xi}, \frac{\partial(\sigma_E-\sigma_I)}{\partial \Xi}(\widehat{\Xi})\Big) \\ - \frac{\partial \Psi}{\partial \Xi}(\widehat{\Xi}, -\frac{\partial \Sigma}{\partial \Xi}(\widehat{\Xi}))\cdot(\Xi-\widehat{\Xi}) - \frac{\partial \Psi}{\partial \tau}\Big(\widehat{\Xi}, -\frac{\partial \Sigma}{\partial \Xi}(\widehat{\Xi})\Big)\cdot\Big(\tau - \frac{\partial(\sigma_E-\sigma_I)}{\partial \Xi}(\widehat{\Xi})\Big)\end{aligned}$$

where $\Sigma = \sigma_I - \sigma_E$. By (3.17), (3.18), $\mathcal{E}_r$ has the simple form

$$\mathcal{E}_r = \frac{1}{2}|v-\hat{v}|^2 + \Psi(\Xi, T - \frac{\partial \sigma_I}{\partial \Xi}) - \sigma_E(\widehat{\Xi}) - \frac{\partial \sigma_E}{\partial \Xi}(\widehat{\Xi})\cdot(\Xi-\widehat{\Xi}) \tag{3.19}$$

We now recall the identities: The H-theorem for the relaxation approximation

$$\partial_t\Big(\frac{1}{2}|v|^2 + \Psi(\Xi,\tau)\Big) - \partial_\alpha(v_i S_{i\alpha}) + \frac{1}{\varepsilon}\frac{\partial \Psi}{\partial \tau^A}\,\big(\tau^A - \frac{\partial(\sigma_E-\sigma_I)}{\partial \Xi^A}\big) = 0 \tag{3.20}$$

and the energy equation for the extended elastodynamics system

$$\partial_t\Big(\frac{1}{2}|\widehat{v}|^2 + \sigma_E(\widehat{\Xi})\Big) - \partial_\alpha\Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\widehat{v}_i\Big) = 0 \tag{3.21}$$

Finally we form the difference equations

$$\begin{aligned}\partial_t(v_i-\widehat{v}_i) - \partial_\alpha\left(T^A\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right) = 0, \\ \partial_t(\Xi^A - \widehat{\Xi}^A) - \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\,v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\,\widehat{v}_i\right) = 0\end{aligned}$$

and compute using (3.6) and (3.8) to obtain

$$\begin{aligned}&\partial_t\Big[\widehat{v}_i(v_i-\widehat{v}_i) + \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\,(\Xi^A-\widehat{\Xi}^A)\Big] \\ &\quad - \partial_\alpha\left[\widehat{v}_i\left(T^A\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\right. \\ &\qquad\qquad \left. + \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\,v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\,\widehat{v}_i\right)\right] \\ &= (\partial_t\widehat{v}_i)(v_i-\widehat{v}_i) + \partial_t\left(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)(\Xi^A-\widehat{\Xi}^A) \\ &\quad - \partial_\alpha\widehat{v}_i\left(T^A\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right) \\ &\qquad - \partial_\alpha\left(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\,v_i - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\,\widehat{v}_i\right) \\ &= -\partial_\alpha\left(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)v_i \\ &\quad - \partial_\alpha\widehat{v}_i\Big[T^A\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F}) - \frac{\partial^2 \sigma_E(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B}\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})(\Xi^B - \widehat{\Xi}^B)\Big]\end{aligned}$$

$$=: I \tag{3.22}$$

By rearranging the terms and using the null-Lagrangian property (2.2) we may rewrite I in the form

$$\begin{aligned}
I &= -\partial_\alpha\Big[\widehat{v}_i \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\Big] \\
&\quad - \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)(v_i - \widehat{v}_i) \\
&\quad - (\partial_\alpha \widehat{v}_i)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) - \frac{\partial^2 \sigma_E}{\partial \Xi^A \partial \Xi^B}(\widehat{\Xi})(\Xi^B - \widehat{\Xi}^B)\Big) \\
&\quad - (\partial_\alpha \widehat{v}_i)\left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right) \\
&\quad - (\partial_\alpha \widehat{v}_i)\left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi)\right)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \\
&= -\partial_\alpha\left[\widehat{v}_i \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\right] - Q_1 - Q_2 - Q_3 - L
\end{aligned} \tag{3.23}$$

That is the term I is written as the sum of a divergence term plus the quadratic terms Q_i plus a linear term L that is controlled by the distance from equilibrium.

Combining (3.20), (3.21), (3.22) and (3.23) we arrive at the relative entropy identity

$$\partial_t \mathcal{E}_r - \partial_\alpha \mathcal{F}_{\alpha,r} + \frac{1}{\varepsilon} D = Q_1 + Q_2 + Q_3 + L \tag{3.24}$$

where the flux is

$$\mathcal{F}_{\alpha,r} := \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)(v_i - \widehat{v}_i)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \tag{3.25}$$

the dissipation is

$$\frac{1}{\varepsilon} D = \frac{1}{\varepsilon}\frac{\partial \Psi}{\partial \tau^A}\Big(\Xi, T - \frac{\partial \sigma_I}{\partial \Xi}\Big)\,(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}) \tag{3.26}$$

the quadratic errors Q_i are

$$\begin{aligned}
Q_1 &= \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)(v_i - \widehat{v}_i) \\
Q_2 &= (\partial_\alpha \widehat{v}_i)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) - \frac{\partial^2 \sigma_E(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B}(\Xi^B - \widehat{\Xi}^B)\Big) \\
Q_3 &= (\partial_\alpha \widehat{v}_i)\left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)
\end{aligned} \tag{3.27}$$

and the linear error L is

$$L = (\partial_\alpha \widehat{v}_i)\left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi)\right)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \tag{3.28}$$

Identity (3.24) is the key on which the stability and convergence analysis of section 4 is based.

4. Stability theorem

Consider a family of smooth solutions $\{(v^\varepsilon, F^\varepsilon, \tau^\varepsilon)\}_{\varepsilon>0}$, $\tau^\varepsilon = T^\varepsilon - \nabla_\Xi \sigma_I(\Phi(F^\varepsilon))$, to the relaxation system (3.1). We wish to compare them with a smooth solution $(\widehat{v}, \widehat{F})$ of the equations of polyconvex elastodynamics (3.2). For simplicity of notation, we drop the ε-dependence from the solution of the relaxation system. The data F_0 and $\widehat{F}_0$ are taken gradients; this property is propagated by $(3.2)_2$ and both F and $\widehat{F}$ are gradients for all times. The function $(v, \Phi(F), \tau)$ is a smooth solution of the augmented relaxation system (3.6) while the function $(\widehat{v}, \Phi(\widehat{F}))$ satisfies the extended elastodynamics equations (3.8). From the results of section 3.3, smooth solutions of (3.6) and (3.8) satisfy (3.24).

The identity (3.24) is inherited by $(v, \Phi(F), \tau)$ and $(\widehat{v}, \Phi(\widehat{F}))$. The resulting relative energy and associated flux,

$$\begin{aligned} e_r &= \mathcal{E}_r\Big(v, \Phi(F), \tau \mid \widehat{v}, \Phi(\widehat{F}), \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi}(\Phi(\widehat{F}))\Big) \\ &= \frac{1}{2}|v - \widehat{v}|^2 + \Psi\Big(\Phi(F), T - \frac{\partial \sigma_I}{\partial \Xi}(\Phi(F))\Big) - \sigma_E(\Phi(\widehat{F})) \\ &\qquad - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(\widehat{F}))(\Phi(F)^A - \Phi(\widehat{F})^A)\,, \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_\alpha &= \mathcal{F}_{\alpha,r}\Big(v, \Phi(F), \tau \mid \widehat{v}, \Phi(\widehat{F}), \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi}(\Phi(\widehat{F}))\Big) \\ &= \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(\widehat{F}))\right)(v_i - \widehat{v}_i)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\,, \end{aligned} \tag{4.2}$$

satisfy

$$\partial_t e_r - \partial_\alpha f_\alpha + \frac{1}{\varepsilon} D = Q_1 + Q_2 + Q_3 + L \tag{4.3}$$

where the Q_i, L and D are now computed for $\Xi = \Phi(F)$ and $\widehat{\Xi} = \Phi(\widehat{F})$.

We prove convergence of the relaxation system to polyconvex elastodynamics so long as the limit solution is smooth.

Theorem 4.1. *Let $(v^\varepsilon, F^\varepsilon, T^\varepsilon)$, $F^\varepsilon = \nabla y^\varepsilon$, be smooth solutions of* (3.1) *and $(\widehat{v}, \widehat{F})$, $\widehat{F} = \nabla \widehat{y}$, a smooth solution of* (3.2)*, defined on $\mathbb{R}^d \times [0, T]$ and decaying fast as $|x| \to \infty$. The relative energy e_r defined in* (4.1) *satisfies* (4.3)*. Assume that σ_I, σ_E satisfy for some constants $\gamma_I > \gamma_v > 0$ and $M > 0$ the hypotheses*

$$\nabla^2 \sigma_I \geq \gamma_I \mathbb{I} > \gamma_v \mathbb{I} \geq \nabla^2(\sigma_I - \sigma_E) > 0\,, \tag{h$_1$}$$

$$|\nabla^2 \sigma_E| \leq M\,, \quad |\nabla^3 \sigma_E| \leq M\,. \tag{h$_2$}$$

There exists a constant s and $C = C(T, \gamma_I, \gamma_v, M, \nabla \widehat{v}, \nabla \widehat{F}) > 0$ independent of ε such that

$$\int_{\mathbb{R}^d} e_r(x,t)dx \leq C\left(\int_{\mathbb{R}^d} e_r(x,0)dx + \varepsilon\right).$$

In particular, if the data satisfy

$$\int_{\mathbb{R}^d} e_r^\varepsilon(x,0)dx \longrightarrow 0\,, \quad \text{as } \varepsilon \downarrow 0\,,$$

then

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|v^\varepsilon-\widehat{v}|^2+|F^\varepsilon-\widehat{F}|^2+|\tau^\varepsilon-\tau_\infty(\widehat{F})|^2dx\longrightarrow 0\,,$$

where $\tau_\infty(\widehat{F})=\frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi}(\Phi(\widehat{F}))$.

Proof. The equation (4.3),

$$\partial_t e_r+\partial_\alpha f_\alpha+\frac{1}{\varepsilon}D=J\,,$$

is integrated on $\mathbb{R}^d\times(0,t)$ and gives

$$\begin{aligned}\int_{\mathbb{R}^d}e_r(x,t)dx-\int_{\mathbb{R}^d}e_r(x,0)dx&\\ +\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^d}Dd x d\tau=\int_0^t\int_{\mathbb{R}^d}Jdxd\tau&\end{aligned}\tag{4.4}$$

From lemma 3.2 and (3.19) we see that there exists a positive constant $c=c(\gamma_I,\gamma_v)$ such that

$$\mathcal{E}_r\geq c\left(|v-\widehat{v}|^2+|\Xi-\widehat{\Xi}||^2+|\tau-\frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi}(\widehat{\Xi})|^2\right)$$

and thus, by (4.1),

$$e_r\geq c\big(|v-\widehat{v}|^2+|\Phi(F)-\Phi(\widehat{F})|^2+|\tau-\tau_\infty(\widehat{F})|^2\big)\,.$$

Note that

$$\begin{aligned}D&:=\frac{\partial\Psi}{\partial\tau_A}\Big(\tau^A-\frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi^A}\Big)\\&=(\Xi+\nabla_\tau G)\cdot(\tau+\nabla_\Xi\Sigma)\\&=(\nabla_\tau G(\tau)-\nabla_\tau G(-\nabla_\Xi\Sigma))\cdot(\tau+\nabla_\Xi\Sigma)\\&\geq\big(\min\nabla^2_\tau G\big)\,|\tau+\nabla_\Xi\Sigma|^2\\&\geq\frac{1}{\gamma_v}|\tau-\nabla_\Xi(\sigma_E-\sigma_I)|^2\end{aligned}\tag{4.5}$$

Let now C be a positive constant depending on the L^∞-norm of $\widehat{v}$, $\widehat{F}$, $\partial_\alpha\widehat{v}$, $\partial_\alpha\widehat{F}$ and the constants γ_I, γ_v and M. On account of (h$_2$) and the smoothness of $(\widehat{v},\widehat{F})$, the term Q_2 is of quadratic growth on $|\Xi-\widehat{\Xi}|=|\Phi(F)-\Phi(\widehat{F})|$. Using (3.27), (h$_2$), and (3.28) we have

$$\begin{aligned}\int_{\mathbb{R}^d}|Q_1|dx&\leq C\int_{\mathbb{R}^d}|v-\widehat{v}|^2+\Big|\frac{\partial\Phi}{\partial F}(F)-\frac{\partial\Phi}{\partial F}(\widehat{F})\Big|^2dx\,,\\ \int_{\mathbb{R}^d}|Q_2|dx&\leq C\int_{\mathbb{R}^d}|\Phi(F)-\Phi(\widehat{F})|^2dx\,,\\ \int_{\mathbb{R}^d}|Q_3|dx&\leq C\int_{\mathbb{R}^d}|\Phi(F)-\Phi(\widehat{F})|^2+\Big|\frac{\partial\Phi}{\partial F}(F)-\frac{\partial\Phi}{\partial F}(\widehat{F})\Big|^2dx\,,\end{aligned}$$

and

$$\int_{\mathbb{R}^d}|L|dx\leq\frac{1}{\varepsilon}\frac{1}{2\gamma_v}\int_{\mathbb{R}^d}|\tau-\nabla_\Xi(\sigma_E-\sigma_I)|^2dx+C\varepsilon\int_{\mathbb{R}^d}|\frac{\partial\Phi}{\partial F}(F)|^2dx\,.$$

From the identities

$$\frac{\partial \det F}{\partial F_{i\alpha}} = (\operatorname{cof} F)_{i\alpha}\,, \qquad \frac{\partial (\operatorname{cof} F)_{i\alpha}}{\partial F_{j\beta}} = \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}F_{k\gamma}\,,$$

we have

$$\Big|\frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F})\Big| \le C|\Phi(F) - \Phi(\widehat{F})|\,.$$

Combining with (4.5) and (4.4) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} e_r(x,t)dx &+ \frac{1}{2\varepsilon\gamma_v}\int_{\mathbb{R}^d} |\tau - \nabla\Xi(\sigma_E - \sigma_I)|^2 dx \\ &= \int_{\mathbb{R}^d} e_r(x,0)dx + C\int_0^t\int_{\mathbb{R}^d} e_r(x,\tau)dxd\tau \\ &\quad + \varepsilon C\int_0^t\int_{\mathbb{R}^d} |\frac{\partial \Phi}{\partial F}(F)|^2 dxd\tau \end{aligned} \tag{4.6}$$

The H-estimate implies that solution of the relaxation system (3.1) satisfy the uniform (in ε) bounds

$$\begin{aligned}\int_{\mathbb{R}^d} |v|^2 + |\Phi(F)|^2 + |\tau|^2 dx &+ \frac{1}{\varepsilon\gamma_v}\int_{\mathbb{R}^d} |\tau - \nabla\Xi(\sigma_E - \sigma_I)|^2 dx \\ &\le C\int_{\mathbb{R}^d} |v_0|^2 + \Psi(\Phi(F_0),\tau_0)dx \le O(1) \end{aligned} \tag{4.7}$$

The result then follows from (4.6) via Gronwall's inequality. □

5. Gas dynamics in Eulerian coordinates

As an example we work out the relaxation model that results when applying (3.1) to the equations of isentropic gas dynamics. In preparation, we review the classical transformation of a balance law from Lagrangean to Eulerian coordinates (*e.g.* [13, Sec 2.2]).

5.1. **Transformation from Lagrangean to Eulerian coordinates.** Consider a motion $y(\cdot,t) : \mathcal{R} \to \mathcal{R}_t$ that maps a reference configuration $\mathcal{R}$ onto the current configuration $\mathcal{R}_t$, for each $t \in [0,T]$. The Lagrangean coordinates in the reference configuration are denoted by $x = (x_\alpha)_{\alpha=1,\dots,d}$ and the Eulerian coordinates in the current configuration by $y = (y_j)_{j=1,\dots,d}$ with d the (common) dimension of the ambient spaces. The map $y(\cdot,t)$ is assumed globally invertible and a bi-Lipschitz homeomosphism, and we denote by

$$v_i = \frac{\partial y_i}{\partial t}\,, \quad F_{i\alpha} = \frac{\partial y_i}{\partial x_\alpha}$$

the velocity and deformation gradient respectively.

Suppose the fields $\phi = \phi(x,t)$, $\psi_\alpha = \psi_\alpha(x,t)$ and $p = p(x,t)$ are defined in the Lagrangian frame and satisfy the balance law

$$\partial_t \phi(x,t) = \partial_\alpha \psi_\alpha(x,t) + p(x,t)\,. \tag{5.1}$$

The fields ϕ, ψ_α and p can be scalar or vector fields. The Lagrangian balance law (5.1) can be transformed to an equivalent balance law expressed on the Eulerian coordinate frame

$$\partial_t\left(\frac{\phi}{\det F}\circ y^{-1}\right) + \partial_{y_j}\left((\frac{\phi}{\det F}v_j)\circ y^{-1}\right) = \partial_{y_j}\left((\frac{\psi_\alpha F_{j\alpha}}{\det F})\circ y^{-1}\right) + \frac{p}{\det F}\circ y^{-1}\,. \tag{5.2}$$

In expressing (5.2) we have used $y^{-1}(y,t)$ to be the inverse (in x) map of $y(x,t)$. This dependence is often implied when stating the balance law in Eulerian coordinates and it is commonplace to write (5.2), using a somewhat ambivalent notation, in the form

$$\partial_t\left(\frac{\phi}{\det F}\right)+\partial_{y_j}\left(\frac{\phi}{\det F}u_j\right)=\partial_{y_j}\left(\frac{\psi_\alpha F_{j\alpha}}{\det F}\right)+\frac{p}{\det F}\,, \tag{5.3}$$

where $u_j = v_j \circ y^{-1}$ (or equivalently $u_j(y(x,t),t) = v_j(x,t)$) stands for the velocity expressed in Eulerian coordinates.

5.2. **Expression of gas dynamics in Lagrangean coordinates.** Consider now the system of isentropic gas dynamics in Eulerian coordinates

$$\partial_t\rho + \partial_j(\rho u_j) = 0 \tag{5.4}$$

$$\partial_t(\rho u_i) + \partial_j(\rho u_i u_j) + \partial_i p(\rho) = 0 \tag{5.5}$$

where $\rho = \rho(y,t)$ the density, $u = u(y,t)$ the velocity, $y \in \mathbb{R}^3$, and the pressure $p(\rho) > 0$ satisfies $p'(\rho) > 0$ which guarantees hyperbolicity. The system of isentropic gas dynamics satisfies the energy conservation equation

$$\partial_t(\frac{1}{2}\rho|u|^2+\rho e(\rho))+\partial_j\big(u_j(\frac{1}{2}\rho|u|^2+\rho e(\rho))\big)+\partial_j(p(\rho)u_j)=0 \tag{5.6}$$

where the internal energy function $e(\rho)$ is related to the pressure through the usual relation

$$e'(\rho)=\frac{p(\rho)}{\rho^2}>0\,. \tag{5.7}$$

Note that $(\rho e)'' = \frac{p'}{\rho} > 0$ and that $\eta(\rho,m) = \frac{1}{2}\frac{|m|^2}{\rho} + \rho e(\rho)$ is convex in the variables (ρ, m), $m = \rho u$ the momentum.

We proceed to calculate the associated Lagrangian form of the system (5.4)-(5.5). For the velocity field $u(y,t)$ the initial value problem

$$\begin{cases}\partial_t y_i = u_i(y,t)\\ y(x,0)=x \qquad x\in\mathcal{R}\end{cases} \tag{5.8}$$

determines the motion $y(x,t)$. The local solvability of (5.8) is guaranteed for sufficiently smooth vector fields u but the solution is not necessarily globally well defined. Here we will not discuss these important aspects and will proceed formally. Given $y(x,t)$ we define $F = \nabla y$, $v = \partial_t y$ and recall Abel's formula

$$\partial_t \det F = \mathrm{div}_y u \det F\,.$$

Using the correspondence between the Lagrangean (5.1) and Eulerian (5.3) form of the balance law, we transform the balance of mass equation (5.4) to the form

$$\partial_t(\rho\det F)=0$$

This implies that $\rho\det F =: \rho_0(x)$ is independent of time. By assigning the reference density of the current configuration (here selected as the $t = 0$ instance of the current configuration) to be $\rho_0(x) = 1$, we obtain

$$\rho=\frac{1}{\det F}\,. \tag{5.9}$$

In turn, using the relations

$$(F^{-1})_{\alpha i} F_{j\alpha} = \delta_{ij} \,, \quad (F^{-1})_{\alpha i} = \frac{1}{\det F} (\operatorname{cof} F)_{\alpha i} \,,$$

and (5.9), (5.5) is transformed into the Lagrangian form

$$\partial_t v_i = \partial_\alpha \big(-p(\rho)(\det F)(F^{-1})_{\alpha i} \big) = \partial_\alpha \big(-p(\rho)(\operatorname{cof} F)_{i\alpha} \big) \tag{5.10}$$

Note the correspondence with the standard definition of the Cauchy stress for gas dynamics $T_{ij} = -p(\rho)\delta_{ij}$ and its association with the Piola-Kirchhoff stress

$$S_{i\alpha} = T_{ij}(\det F)(F^{-1})_{\alpha j} = -p(\rho)(\det F)(F^{-1})_{\alpha i}$$

Similarly, the energy equation (5.6) transforms to the Lagrangean form

$$\partial_t \big(\frac{1}{2}|v|^2 + e(\rho) \big) = \partial_\alpha \big(-p(\rho) v_i (\operatorname{cof} F)_{i\alpha} \big)$$

To the above equations we may add the nonlinear transport relation

$$\partial_t \det F = \partial_\alpha (v_i (\operatorname{cof} F)_{i\alpha})$$

which is a consequence of the null Lagrangians (2.2) and part of (2.3).

In summary, the full set of Lagrangean equations for gas dynamics is

$$\partial_t F_{i\alpha} = \partial_\alpha v_i \tag{5.11}$$

$$\partial_t \det F = \partial_\alpha (v_i (\operatorname{cof} F)_{i\alpha}) \tag{5.12}$$

$$\partial_t v_i = \partial_\alpha \Big(-p\Big(\frac{1}{\det F}\Big) (\operatorname{cof} F)_{i\alpha} \Big) \tag{5.13}$$

and the Lagrangean form of the energy is

$$\partial_t \Big(\frac{1}{2}|v|^2 + e\Big(\frac{1}{\det F}\Big) \Big) = \partial_\alpha \Big(-p\Big(\frac{1}{\det F}\Big) v_i (\operatorname{cof} F)_{i\alpha} \Big) \,. \tag{5.14}$$

The stored energy W is of the form

$$W(F) = e\Big(\frac{1}{\det F}\Big) = g(\det F) \tag{5.15}$$

where

$$\begin{aligned} g(w) &:= e\left(\frac{1}{w}\right) \\ \frac{dg}{dw} &= e'\Big(\frac{1}{w}\Big)\Big(-\frac{1}{w^2}\Big) = -p\Big(\frac{1}{w}\Big) \,, \\ \frac{d^2 g}{dw^2} &= p'\Big(\frac{1}{w}\Big)\frac{1}{w^2} > 0 \,, \end{aligned}$$

Hence W is polyconvex, the system (5.11)-(5.13) fits into the framework of polyconvex elasticity with the identification $g(w) := e\left(\frac{1}{w}\right)$, and of course it is associated with an extended symmetrizable system of the form (2.4) for the variables (F, Ξ) with $\Xi = (F, w)$ in the present case.

5.3. **A relaxation model for gas dynamics in Lagrangean coordinates.** We consider now the relaxation model

$$\begin{aligned} \partial_t v_i &= \partial_\alpha \Big(\big[- p_I\big(\frac{1}{\det F}\big) + \tau \big] \operatorname{cof} F_{i\alpha} \Big) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t \tau &= -\frac{1}{\varepsilon} \Big(\tau + p_E\big(\frac{1}{\det F}\big) - p_I\big(\frac{1}{\det F}\big) \Big) \end{aligned} \tag{5.16}$$

This model is a special case of the model (3.1) with a scalar internal variable

$$T = -p_I\big(\frac{1}{\det F}\big) + \tau$$

We assume that the instantaneous $p_I(\rho)$ and equilibrium $p_E(\rho)$ pressure functions are strictly positive and satisfy

$$\begin{aligned} p_I'(\rho) > 0\,, \quad e_I'(\rho) = \frac{p_I(\rho)}{\rho^2} \\ p_E'(\rho) > 0\,, \quad e_E'(\rho) = \frac{p_E(\rho)}{\rho^2} \end{aligned} \tag{a$_0$}$$

with $e_I(\rho)$ and $e_E(\rho)$ the associated instantaneous and equilibrium internal energy functions.

Furthermore, (5.16) is associated with the augmented relaxation system (*cf* (3.6))

$$\begin{aligned} \partial_t v_i &= \partial_\alpha \Big(\big[- p_I\big(\frac{1}{w}\big) + \tau \big] \operatorname{cof} F_{i\alpha} \Big) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t w &= \partial_\alpha \big((\operatorname{cof} F)_{i\alpha} v_i \big) \\ \partial_t \tau &= -\frac{1}{\varepsilon} \Big(\tau + p_E\big(\frac{1}{w}\big) - p_I\big(\frac{1}{w}\big) \Big) \end{aligned} \tag{5.17}$$

with $w > 0$, and the theory developed in section 3 can be applied directly to (5.17) with the following identifications

$$\begin{aligned} \sigma_I(w) = e_I\big(\frac{1}{w}\big)\,, \quad \frac{d\sigma_I}{dw} = -p_I\big(\frac{1}{w}\big) \\ \sigma_E(w) = e_E\big(\frac{1}{w}\big)\,, \quad \frac{d\sigma_E}{dw} = -p_E\big(\frac{1}{w}\big) \end{aligned}$$

where by (a$_0$) both $\sigma_I(w)$ and $\sigma_w(w)$ are convex.

Following the procedure of section 3.2, we obtain an entropy for the augmented relaxation system and in turn for the reduced system (5.16). By multiplying $(5.17)_1$ by v_i, $(5.17)_3$ by $\big(- p_I(\frac{1}{w}) + \tau \big)$, and $(5.17)_4$ by $(w + G'(\tau))$ we obtain the entropy equation

$$\begin{aligned} \partial_t \Big(\frac{1}{2}|v|^2 + e_I\big(\frac{1}{w}\big) + w\tau + G(\tau) \Big) &= \partial_\alpha \Big(\big[- p_I\big(\frac{1}{w}\big) + \tau \big] (\operatorname{cof} F)_{i\alpha} v_i \Big) \\ &\quad - \frac{1}{\varepsilon} \big(w + G'(\tau) \big) \Big(\tau + (p_E - p_I)\big(\frac{1}{w}\big) \Big) \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} \sigma_I(w) &:= e_I\big(\frac{1}{w}\big) = -\int_1^w p_I\big(\frac{1}{s}\big) ds \\ G(\tau) &:= -\int_1^\tau \frac{1}{(p_I - p_E)^{-1}(s)} ds \end{aligned}$$

Indeed, if the pressure functions satisfy the hypothesis

$$(p_I - p_E)'(\rho) > 0 \quad \forall \rho > 0 \tag{a$_1$}$$

then $(p_I - p_E)^{-1}$ and $G(\tau)$ are well defined, $\Sigma(w) = (\sigma_I - \sigma_E)(w)$ is convex and Lemma 3.1 guarantees the existence of a global, dissipative entropy

$$\Psi(w, \tau) = \sigma_I(w) + w\tau + G(\tau)\,.$$

Using Lemma 3.2 it follows that the entropy $\Psi(w, \tau)$ is convex in (w, τ) provided

$$\frac{p_I'(\frac{1}{w})}{w^2} \geq \frac{(p_I - p_E)'(\frac{1}{\bar{w}})}{\bar{w}^2}\,, \quad \forall w, \bar{w} > 0\,. \tag{a$_2$}$$

5.4. **Expression of the relaxation model in Eulerian coordinates.** We next apply again the transformation procedure from Lagrangean to Eulerian coordinates to express the model (5.16) in Eulerian coordinates. We recall the expression $\rho = \frac{1}{\det F}$ and note that (5.16) when expressed in Eulerian coordinates gives

$$\begin{aligned} \partial_t \rho + \partial_j(\rho u_j) &= 0 \\ \partial_t(\rho u_i) + \partial_j(\rho u_i u_j) &= \partial_j\big((-p_I(\rho) + \tau)\delta_{ij}\big) \\ \partial_t(\rho\tau) + \partial_j(\rho u_j \tau) &= -\frac{1}{\varepsilon}\rho\big(\tau - p_I(\rho) + p_E(\rho)\big) \end{aligned} \tag{5.19}$$

This is a pressure relaxation model with two pressures an instantaneous and an equilibrium pressure. Models of that general type have previously been observed in the literature, see for example [9, 23]. Such models correspond to a mechanism of relaxation of pressures with an instantaneous and an equilibrium pressure response, the latter associated with the long time response of the model in the way outlined in section 3, and are endowed with and entropy function defined locally (near equilibrium) which is dissipative [23]. The present model is endowed with a *globally defined* entropy function. This can be seen by reverting the entropy dissipation identity (5.18) into Eulerian coordinates. The process gives

$$\begin{aligned} &\partial_t \left[\frac{1}{2}\rho|v|^2 + \rho\big(e_I(\rho) + \frac{1}{\rho}\tau + G(\tau)\big)\right] + \partial_j\left(u_j\big[\frac{1}{2}\rho|v|^2 + \rho\big(e_I(\rho) + \frac{1}{\rho}\tau + G(\tau)\big)\big]\right) \\ &\qquad = \partial_j\left((-p_I(\rho) + \tau)u_j\right) - \frac{1}{\varepsilon}\rho\big(\tau - (p_I - p_E)(\rho)\big)\left(\frac{1}{\rho} - \frac{1}{(p_I - p_E)^{-1}(\tau)}\right) \end{aligned} \tag{5.20}$$

The existence of globally defined entropy relaxation functions is noted in [10] in the context of internal energy relaxation models for gas dynamics and for general models with internal variables in [24]. The present model provides another example that enjoys this feature and is associated with pressure relaxation and is related to the polyconvexity property of the elasticity system.

We finish by checking the conditions under which the above expressions are well defined. It is instructive to check that directly. We always operate under the framework of (a$_0$) and let $P(\rho) = (p_I - p_E)(\rho)$. The entropy will be dissipative provided

$$\rho(\tau - P(\rho))\Big(\frac{1}{\tau} - \frac{1}{P^{-1}(\rho)}\Big) \geq 0\,, \quad \forall \rho, \tau > 0\,. \tag{5.21}$$

The equation (5.21) holds if and only if the function P is strictly increasing, that is if (a$_1$) holds.

Next, we examine the convexity of the function

$$\rho E(\rho,\tau) := \rho\left(e_I(\rho) + \frac{1}{\rho}\tau + G(\tau)\right) \tag{5.22}$$

by checking the eigenvalues of the Hessian matrix

$$\begin{pmatrix} \frac{d^2}{d\rho^2}(\rho e_I) & G'(\tau) \\ G'(\tau) & \rho G''(\tau) \end{pmatrix}.$$

The eigenvalues are strictly positive if

$$(\rho e_I)'' = \frac{p_I'}{\rho} > 0\,,$$
$$(\rho e_I)''\rho G''(\tau) - (G'(\tau))^2 > 0\,.$$

To express the second condition, note that if $\tau = P(\bar\rho)$ then $\bar\rho = P^{-1}(\tau)$ and

$$G'(\tau) = -\frac{1}{P^{-1}(\tau)} = -\frac{1}{\bar\rho}\,, \quad G''(\tau) = \frac{1}{[P^{-1}(\tau)]^2}\frac{1}{P'(P^{-1}(\tau))} = \frac{1}{\bar\rho^2 P'(\bar\rho)}$$

In view of (a_0), the convexity of $\rho E(\rho,\tau)$ is equivalent to the condition

$$p_I'(\rho) > (p_I' - p_E')(\bar\rho) > 0\,, \qquad \forall \rho,\bar\rho > 0\,. \tag{a_3}$$

This can be combined with the fact that the function $\frac{|m|^2}{\rho}$ is convex in (ρ, m) to conclude that the under (a_3) the entropy

$$H(\rho,\tau,m) := \frac{1}{2}\frac{|m|^2}{\rho} + \rho\big(e_I(\rho) + \frac{1}{\rho}\tau + G(\tau)\big) \tag{5.23}$$

is convex in (ρ,τ,m) with $m = \rho u$ the momentum.

Acknowledgements. Partially supported by the EU FP7-REGPOT project "Archimedes Center for Modeling, Analysis and Computation" and by the "Aristeia" program of the Greek Secretariat for Research.

References

[1] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337-403.

[2] J.M. Ball, Some open problems in elasticity In: *Geometry, mechanics and dynamics*, Springer, New York, 2002, pp. 3-59.

[3] F. Bouchut, *Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources.* Frontiers in Mathematics. Birkhuser Verlag, Basel, 2004.

[4] Y. Brenier, Hydrodynamic structure of the augmented Born-Infeld equations, *Arch. Rational Mech. Anal.* **172** (2004), 65-91.

[5] F. Bouchut, C. Klingenberg, and K. Waagan, A multiwave approximate Riemann solver for ideal MHD based on relaxation. I. Theoretical framework. *Numer. Math.* **108** (2007), 742.

[6] P.G. Ciarlet, *Mathematical Elasticity*, Vol. 1, North Holland, (1993).

[7] G.-Q. Chen, C.D. Levermore and T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure and Appl. Math.* **47** (1994), 787–830.

[8] B. Coleman and W. Noll, On the Thermostatics of Continuous Media. *Arch. Rat. Mech. Anal.*, **4** (1959), 97-128.

[9] B.D. Coleman and M.E. Gurtin, Thermodynamics with internal state variables, *J. Chem. Physics* **47** (1967), 597-613.

[10] F. Coquel and B. Perthame, Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, *SIAM J. Num. Anal.* **35** (1998), 2223-2249.
[11] C.M. Dafermos, The second law of thermodynamics and stability, *Arch. Rational Mech. Analysis* **70** (1979), 167-179.
[12] C.M. Dafermos, Quasilinear hyperbolic systems with involutions, *Arch. Rational Mech. Anal.* **94** (1986), 373-389.
[13] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd edition, Grundlehren der Mathematischen Wissenschaften, vol. 325, Springer-Verlag, Berlin, 2010.
[14] S. Demoulini, D.M.A. Stuart and A.E. Tzavaras, A variational approximation scheme for three–dimensional elastodynamics with polyconvex energy, *Arch. Ration. Mech. Anal.* **157** (2001), 325–344.
[15] S. Demoulini, D.M.A. Stuart and A.E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics, *Arch. Ration. Mech. Anal.* **205** (2012), 927-961.
[16] R.J. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* **28** (1979), 137-188.
[17] C. Faciu and M. Mihailescu-Suliciu, The energy in one-dimensional rate-type semilinear viscoelasticity, *Int. J. Solids Structures* **23** (1987), 1505-1520.
[18] C. Lattanzio and A. Tzavaras, Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics. *Arch. Rational Mech. Anal.* **180** (2006), 449-492.
[19] P.D. Lax, Shock waves and entropy, in: "Contributions to Nonlinear Functional Analysis." E.H. Zarantonello, ed. New York: Academic Press, 1971, pp. 603-634.
[20] T. Qin, Symmetrizing nonlinear elastodynamic system, *J. Elasticity* **50** (1998), 245-252.
[21] D. Serre, Hyperbolicity of the Nonlinear Models of Maxwell's Equations, *Arch. Rational Mech. Anal.* **172** (2004), 309–331.
[22] M. Slemrod, *Lectures on the Isometric Embedding Problem* $(M^n, g) \to \mathbb{R}^m$*,* $m = \frac{n}{2}(n+1)$, Unpublished Lecture Notes, 2013.
[23] I. Suliciu, On the thermodynamics of rate-type fluids and phase transitions. I. Rate-type fluids, *Internat. J. Engrg. Sci.* **36** (1998), no. 9, 921–947.
[24] A.E. Tzavaras, Materials with internal variables and relaxation to conservation Laws, *Arch. Rational Mech. Anal.* **146** (1999), 129–155.
[25] A.E. Tzavaras, Relative entropy in hyperbolic relaxation *Comm. Math. Sci* **3** (2005), 119-132.
[26] A. Tzavaras, A relaxation theory with polyconvex entropy function converging to elastodynamics. (2005), (unpublished).

Department of Applied Mathematics, University of Crete, Heraklion, Greece and Institute for Applied and Computational Mathematics, FORTH, Heraklion, Greece
E-mail address: `tzavaras@tem.uoc.gr`