SIAM J. MATH. ANAL.
Vol. 30, No. 6, pp. 1309–1342

REPRESENTATION OF WEAK LIMITS AND DEFINITION OF NONCONSERVATIVE PRODUCTS*

PHILIPPE G. LEFLOCH† AND ATHANASIOS E. TZAVARAS‡

Abstract. The goal of this article is to show that the notion of generalized graphs is able to represent the limit points of the sequence $\{g(u_n)\,du_n\}$ in the weak-$\star$ topology of measures when $\{u_n\}$ is a sequence of continuous functions of uniformly bounded variation. The representation theorem induces a natural definition for the nonconservative product $g(u)\,du$ in a BV context. Several existing definitions of nonconservative products are then compared, and the theory is applied to provide a notion of solutions and an existence theory to the Riemann problem for quasi-linear, strictly hyperbolic systems.

Key words. nonconservative products, hyperbolic systems, shock wave, self-similar solution

AMS subject classifications. Primary, 35L60; Secondary, 28A75

PII. S0036141098341794

1. Introduction.

The objective of this article is to present a theoretical frame for the definition and properties of nonconservative products in one space dimension. The issue of defining nonconservative products appears with Volpert's chain rule [31] for BV functions in several space dimensions. It is a central problem for defining a notion of weak solutions for a general quasi-linear hyperbolic system

$$\partial_t u + A(u)\,\partial_x u = 0, \qquad u(x,t) \in \mathbb{R}^N, \quad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

Such systems appear in several models of the engineering and physics literature, e.g., [5, 8, 23, 24, 25, 28]. The origin of the nonconservative terms is usually a simplifying modeling assumption or a closure hypothesis. If (1.1) is conservative, i.e., $A(u) = \nabla F(u)$ for some $F : \mathbb{R}^N \to \mathbb{R}^N$, then weak solutions are defined in the sense of distributions. In the general case, however, the term $A(u)\,\partial_x u$ will contain products of discontinuous functions with measures, and its definition is not obvious. At present, successful definitions exist in the one-space dimensional BV framework by LeFloch [14, 15], Dal Maso, LeFloch, and Murat [10] and Raymond [27]. The definition in [10] is based on a family of Lipschitz paths, is stable under weak convergence, and leads to a solution of the Riemann problem in the class of genuinely nonlinear, strictly hyperbolic systems with Riemann data that are sufficiently close. It has prompted investigations on existence of weak solutions to (1.1), LeFloch and Liu [17], and on convergence of numerical schemes, Hou and LeFloch [12]. The concept of extended graphs is used in [27] to provide a general definition that is stable under weak convergence.

*Received by the editors July 13, 1998; accepted for publication February 15, 1999; published electronically October 8, 1999. This work was supported in part by TMR project HCL ERBFM-RXCT960033.

http://www.siam.org/journals/sima/30-6/34179.html

†Centre de Mathématiques Appliquées and Centre National de la Recherche Scientifique, UA 756, Ecole Polytechnique, 91128 Palaiseau, France (lefloch@cmap.polytechnique.fr). The research of this author was partially supported by the Centre National de la Recherche Scientifique and by the National Science Foundation via a Faculty Early Career Development award (CAREER).

‡Department of Mathematics, University of Wisconsin, Madison, WI 53706, and Institute of Applied and Computational Mathematics, FORTH, 711 10 Heraklion, Crete (tzavaras@math.wisc.edu). This research was completed while this author was visiting the Centre de Mathématiques Appliquées of Ecole Polytechnique and was partially supported by the National Science Foundation and the Office of Naval Research.

Related issues appear in studies of transport equations with discontinuous coefficients (e.g., LeFloch [15, 16], LeFloch and Xin [20], Poupaud and Rascle [22], Bouchut and James [4]) and in minimization of certain types of functionals in the space of functions of bounded variation (e.g., Aviles and Giga [1], Raymond and Seghir [26]). The reader is referred to Colombeau [6], Colombeau and Leroux [7] for a theory of nonconservative products in a weaker functional framework.

Let $g : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function and $u : [a, b] \to \mathbb{R}^N$ be a function of bounded variation. Our scope is to provide a justifiable definition for the inner product of $g(u)$ and $\frac{du}{dx}$, formally given by $g(u)\frac{du}{dx} = \sum_{i=1}^N g^i(u)\,\frac{du^i}{dx}$. This definition will be suggested by a representation theory of the limit points of sequences $\{g(u_n)\frac{du_n}{dx}\}$ in weak topologies when the functions u_n are *smooth*. This viewpoint reflects the premise that (1.1) arises in the limit of regularized problems as the dissipative mechanisms, such as viscosity or relaxation time, tend to zero. Accordingly, the nonconservative product will appear as a limit of regularized sequences.

If u is a *continuous* BV function, there is a natural definition of the product $\mu = g(u)\,\frac{du}{dx}$ as a Radon measure on $\mathcal{C}[a, b]$. This is done by setting

$$\langle \mu, \theta\rangle = \int_{[a,b]} \theta(x) g(u(x))\, du(x), \quad \theta \in \mathcal{C}[a, b], \tag{1.2}$$

where the right-hand side is viewed as a Borel–Stieltjes integral relative to the (vector-valued, signed) measure generated by $u \in \mathcal{C}\cap BV$. This definition is appropriate when u is continuous. If u has discontinuities, definition (1.2) is “not stable,” because the integral $\int f du$, for $f \in L^1(du)$, changes values when changing f at the points of discontinuity of u.

Consider a sequence $\{u_n\}$ of continuous functions $u_n : [a, b] \to \mathbb{R}^N$ that are of uniformly bounded variation

$$\sup_{[a,b]} |u_n| + TV_{[a,b]}\,(u_n) \le C. \tag{1.3}$$

The products $g(u_n)du_n$ are well defined by (1.2) and belong to $\mathcal{M}[a, b] = \big[\mathcal{C}[a, b]\big]^\star$, the dual space of $\mathcal{C}\big([a, b]; \mathbb{R}^N\big)$. The space of Radon measures $\mathcal{M}[a, b]$ is usually equipped either with the strong topology, generated by the dual norm $\|.\|_{\mathcal{M}}$, or with the weak-$\star$ topology. On account of (1.3), the sequence $\{g(u_n)du_n\}$ satisfies $\|g(u_n)du_n\|_{\mathcal{M}} \le C'$. (Throughout $C, C', \dots$ will stand for constants that are independent of n.) Therefore, along a subsequence,

$$g(u_{n'})\,\frac{du_{n'}}{dx} \rightharpoonup \mu \quad \text{weak-}\star \text{ in } \mathcal{M}[a, b] \tag{1.4}$$

to some measure μ. Example 1.1 illustrates that, even if $u_n(x) \to u(x)$ pointwise, the sequence $\{g(u_n)du_n\}$ may have multiple limit points in the weak-$\star$ topology.

Example 1.1. Let u_0, u_1 be two states in $\mathbb{R}^N$, $x_0 \in (a, b)$ and $\pi : [0, 1] \to \mathbb{R}^N$ be a Lipschitz continuous path satisfying $\pi(0) = u_0$ and $\pi(1) = u_1$. Consider the sequence of functions v_n, defined by

$$v_n(x) := \begin{cases} u_0 & \text{if } x \in [a, x_0 - 1/n], \\ \pi\big(\frac{x-(x_0-1/n)}{2/n}\big) & \text{if } x \in [x_0 - 1/n, x_0 + 1/n], \\ u_1 & \text{if } x \in [x_0 + 1/n, b]. \end{cases} \tag{1.5}$$

As $n \to \infty$, the sequence $\{v_n\}$ converges pointwise,

$$v_n(x) \to v(x) := \begin{cases} u_0 & \text{if } x \in [a, x_0), \\ \pi(1/2) & \text{if } x = x_0, \\ u_1 & \text{if } x \in (x_0, b], \end{cases} \tag{1.6}$$

and a calculation shows that

$$\int_a^b \varphi(x) g(v_n(x)) \frac{dv_n}{dx}\, dx \to \left(\int_0^1 g(\pi(s))\pi'(s) ds \right) \varphi(x_0)$$

for $\varphi \in \mathcal{C}[a, b]$. That is,

$$g(v_n)\frac{dv_n}{dx} \rightharpoonup c(g, \pi)\delta_{x_0} \quad \text{weak-}\star \text{ in } \mathcal{M}[a, b], \tag{1.7}$$

where δ_{x_0} stands for the Dirac measure at x_0 and the scalar $c(g, \pi)$ is given by the formula

$$c(g, \pi) := \int_0^1 g(\pi(s))\pi'(s) ds. \tag{1.8}$$

Therefore, first, the limit points of $\{g(v_n)\frac{dv_n}{dx}\}$ depend on the limiting graph selected by $\{v_n\}$, expressed via the path π. Second, by mixing sequences whose internal structure is described by several distinct paths π_j, it is easy to generate a sequence $\{v_n\}$ which converges pointwise to the (same) limit v, but where $\{g(v_n)\frac{dv_n}{dx}\}$ has multiple weak-$\star$ limit points. There exists a notable exception to these features: If $g = \nabla f$ for some $f : \mathbb{R}^N \to \mathbb{R}$, then $c(g, \pi) = f(u_1) - f(u_0)$ and the weak-$\star$ limit (1.7) is independent of π. $\square$

To characterize the weak-$\star$ limit points of $\{g(u_n)du_n\}$ we follow the approach of Tartar [29], in his representation theory of weak limits via Young measures. Let $\mathcal{C}_0([a, b]\times\mathbb{R}^N)$ be the space of $\mathbb{R}^N$-valued continuous functions $f = f(x, \lambda)$ that tend to zero as $\lambda \in \mathbb{R}^N$ tends to infinity, equipped with the sup-norm, and let $\mathcal{M}([a, b]\times\mathbb{R}^N) = \big[\mathcal{C}_0([a, b] \times \mathbb{R}^N)\big]^\star$ be the dual space of Radon measures on $[a, b] \times \mathbb{R}^N$. Define the Radon measures p_n by

$$\langle p_n, f\rangle := \int_a^b f(x, u_n(x))\, du_n(x) \qquad \text{for } f \in \mathcal{C}_0([a, b] \times \mathbb{R}^N). \tag{1.9}$$

Then (1.3) implies that

$$|\langle p_n, f\rangle| \le \big(TV_{[a,b]}\,(u_n)\big) \sup_{x\in[a,b], |\lambda|\le C} |f(x, \lambda)|$$

and, hence, $\|p_n\|_{\mathcal{M}} \le C$. There exist a subsequence $\{p_{n_k}\}$ and a measure $p \in \mathcal{M}([a, b] \times \mathbb{R}^N)$ such that

$$p_{n_k} \rightharpoonup p \quad \text{weakly-}\star \text{ in } \mathcal{M}([a, b] \times \mathbb{R}^N). \tag{1.10}$$

The question becomes to characterize the weak-$\star$ limit points of the sequence $\{p_n\}$.

The characterization is effected by using the concept of *graph completion* or (as we prefer to call it) *generalized graph*. This concept was introduced by Bressan and

Rampazzo [3] in a context of control problems and turns out to be sufficiently discriminating to capture the limiting graphs of the sequence $\{u_n\}$. Generalized graphs were used by Dal Maso, LeFloch, and Murat [10] and Raymond [27] as intermediate steps in their definitions of nonconservative products.

DEFINITION 1.2 (see [3]). *A generalized graph of u is a map $(X, U) : [0,1] \to [a,b] \times \mathbb{R}^N$ such that X, U are Lipschitz continuous and satisfy*

(1) $(X(0), U(0)) = (a, u(a))$, $(X(1), U(1)) = (b, u(b))$;

(2) *X is increasing: $s_1 < s_2$ implies $X(s_1) \leq X(s_2)$;*

(3) *given $y \in [a,b]$, there exists $s \in [0,1]$ such that $X(s) = y$, $U(s) = u(y)$.*

Our aim is to reveal the central role of generalized graphs in providing a geometrically motivated definition of nonconservative products. To this end we exploit an equivalence relation on the space of continuous functions, accounting for reparametrizations of graphs, and the associated pseudometric of uniform graph convergence [3]. By definition, a sequence of graphs $\{gr(u_n)\}$ is Cauchy in the sense of graph convergence, if upon reparametrizing its elements $gr(u_n)$ we obtain a Cauchy sequence in the uniform metric. We will show that, given a sequence of *continuous* functions $\{u_n\}$ that is bounded in $BV[a,b]$, generalized graphs emerge as and are in correspondence to limit points of the sequence of graphs of u_n, $\{gr(u_n)\}$, in the pseudometric of uniform graph convergence. Therefore, the terminology "graph completion" is somewhat misleading, in that it suggests that the completion of the graph is effected arbitrarily from the outside. Since such objects emerge as limits of graphs of sequences of continuous functions, we opt for the more pertinent terminology *generalized graph*. Using this notion we prove a representation theorem on the weak-$\star$ limits in (1.4) and (1.10).

THEOREM 1.3. (a) *Let $\{u_n\}$ be a sequence of continuous functions satisfying the uniform bounds* (1.3). *There exists a subsequence $\{u_{n_k}\}$ and a generalized graph (X, U) such that, for any continuous function $g = g(\lambda)$, we have*

$$\int_{[a,b]} \theta(x) g(u_{n_k}(x))\, du_{n_k}(x) \to \langle \mu(g), \theta \rangle \quad \textit{for} \quad \theta \in \mathcal{C}[a,b], \tag{1.11}$$

where $\mu : \mathcal{C}_0(\mathbb{R}^N) \to \mathcal{M}[a,b]$ is defined by

$$\langle \mu(g), \theta \rangle = \int_0^1 \theta(X(s)) g(U(s))\, dU(s). \tag{1.12}$$

(b) *Conversely, given a generalized graph (X, U), let μ be defined by* (1.12). *There exists a sequence of Lipschitz functions $\{u_n\}$, uniformly bounded in BV, such that for any continuous g,*

$$g(u_n) du_n \rightharpoonup \mu(g) \quad \textit{weak-}\star \textit{ in } \mathcal{M}[a,b]. \tag{1.13}$$

The plan of the article is as follows. Section 2 is preliminary, presenting a change of variable formula for Borel–Stieltjes integrals, an equivalence relation accounting for reparametrizations of continuous paths, and the notion of uniform graph convergence. The case of a continuous BV function is also considered; we introduce the arc-length (or canonical) parametrization of the graph of u and use it, in conjunction with the change of variable formula, to explore the ramifications of definition (1.2) for the nonconservative product $g(u)\, du$, with $u \in \mathcal{C} \cap BV$.

In sections 3 and 4 we study properties of sequences of continuous functions $\{u_n\}$ that are bounded in $BV[a,b]$. After presenting the notion of a generalized graph, we show that, first, generalized graphs arise as limits of subsequences to $\{gr(u_n)\}$ in the pseudometric of graph convergence and, second, that a given generalized graph can always be approximated by a suitable sequence of graphs of continuous functions. The results are summarized in Theorem 3.2 and are put in a metric space framework at the end of section 3.1. Then in section 3.2 we prove a representation theorem.

The representation theorem suggests to define nonconservative products as measures based on generalized graphs. Two definitions, along with associated weak stability theorems, are pursued: In section 4.1, the nonconservative product is defined as a Radon measure (Definition 4.1), while, in section 4.2, it is defined as a signed Borel measure via its distribution function (Definition 4.4). The definitions are equivalent and invariant under reparametrizations of the geometric graph determined by (X, U); i.e., they depend on the equivalence class of the generalized graph (X, U) but not on the specific representative.

In sections 4.3 and 4.4, we compare various definitions of nonconservative products. To assess the issue, it is instructive to keep in mind the analogy to the solution of the Riemann problem for hyperbolic systems. There exist two approaches for solving the Riemann problem: In the first the solution is effected by patching together elementary solutions (shocks, rarefaction waves, and contact discontinuities), while in the second the whole wave fan is visualized to emerge as a single structure in a small parameter (viscosity, relaxation, etc.) limit of a higher-order theory. Accordingly, two viewpoints for defining nonconservative products can be taken: (i) the product is defined in a pointwise fashion by using a predetermined family of paths at points of jump discontinuity, (ii) the product is defined on the whole structure (the generalized graph). The comparison hinges on the relation between generalized graphs and graphs of functions of bounded variation (Propositions 4.7 and 4.8). The emerging definitions are consistent, with each being more adept for a different range of applications. Section 4.4 analyzes several typical examples of nonconservative products.

We complete the article with a study of the Riemann problem for quasi-linear hyperbolic systems. For genuinely nonlinear systems the solution of the Riemann problem is established in LeFloch [14] and Dal Maso, LeFloch, and Murat [10]. The main step is a construction of the shock curves in the nonconservative case, in the spirit of Lax [13]. The present result is based on an entirely different construction process, following the method of self-similar zero-viscosity limits (see Dafermos [9], Tzavaras [30]). It yields a solution for weak waves of the Riemann problem in the class of general strictly hyperbolic systems with no further assumptions (like genuine nonlinearity or finite number of inflection points) on the characteristic fields. The necessary a priori BV estimates are established in the companion articles [18, 19].

2. Preliminary notions.

2.1. Change of variables formula. Throughout, we work in the framework of functions of bounded variation. The total variation of an $\mathbb{R}^N$-valued function u on an interval $[a, b]$ is defined by

$$TV_{[a,b]}(u) := \sup \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})|,$$

where $|\cdot|$ stands for the Euclidean length in $\mathbb{R}^N$ and the supremum is taken over all finite partitions $a = x_0 < x_1 < \cdots < x_n = b$. Let $u : [a, b] \to \mathbb{R}^N$ be a function of bounded variation and let $T_u : [a, b] \to [0, \infty)$ be the total variation function of u, defined by

$$T_u(x) := TV_{[a,x]}(u) \qquad \text{for } x \in [a,b]. \tag{2.1}$$

The domain of u can be decomposed into two disjoint sets: $\mathcal{C}_u$ the set of points of continuity of u and $\mathcal{S}_u$ the set of points of discontinuity, respectively. The set $\mathcal{S}_u$ is at most countable, and the right and left limits $u(x+)$, $u(x-)$, for $x \in (a,b)$, and $u(a+)$, $u(b-)$ exist and are finite. We use the notation $u(a-) = u(a)$ and $u(b+) = u(b)$. Note that x is a point of (right or left) continuity for u if and only if x is a point of (right or left) continuity for T_u. In the particular case that u is Lipschitz continuous (or even when u is absolutely continuous, $u \in W^{1,1}\big(a,b\big)$), the total variation function T_u can be computed by the formula

$$T_u(x) = \int_a^x |u'(y)|\, dy. \tag{2.2}$$

If u is of bounded variation and right continuous on (a,b), there exists a unique finite, signed Borel measure μ_u generated by u,

$$u(x) - u(a+) = \mu_u((a,x]) \ \ \text{for } x \in (a,b], \qquad u(a+) - u(a) = \mu_u(\{a\}).$$

The measure μ_u is typically denoted by du, its total variation measure satisfies $|du| = dT_u$, and it can be decomposed into an absolutely continuous part $u'(x)dx$, an atomic part $d_a u$, and a singular part (relative to the Lebesgue measure) $d_s u$, according to the formula $du = u'(x)dx + d_a u + d_s u$.

For functions u, $v : [a,b] \to \mathbb{R}^N$ right continuous and of bounded variation, there is an integration by parts formula: If u and v have no common points of discontinuity, $\mathcal{S}_u \cap \mathcal{S}_v = \emptyset$, then

$$\int_{[\alpha,\beta]} v(x)du(x) + \int_{[\alpha,\beta]} u(x)dv(x) = v(\beta+)u(\beta+) - v(\alpha-)u(\alpha-) \tag{2.3}$$

for any $[\alpha,\beta] \subset [a,b]$. (Here and in what follows we use the notation $v\,du$ to mean the inner product $\sum_i v_i du_i$, where u_i and v_i are the components of u and v, respectively.) If v is absolutely continuous, (2.3) takes the more conventional form

$$\int_{[\alpha,\beta]} v(x)du(x) = -\int_{[\alpha,\beta]} u(x)v'(x)dx + v(\beta)u(\beta+) - v(\alpha)u(\alpha-). \tag{2.4}$$

We will need certain change of variable formulas that follow from a general measure theoretic construction. We first outline the general construction of image measures, taken out of Folland [11, p. 287]. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space, let $(\Omega', \mathcal{B}')$ be a measurable space, and let $\varphi : \Omega \to \Omega'$ be a $(\mathcal{B}, \mathcal{B}')$-measurable map. Then μ induces an image measure μ^φ on Ω' by

$$\mu^\varphi(E) = \mu\big(\varphi^{-1}(E)\big) \qquad \text{for } E \in \mathcal{B}'. \tag{2.5}$$

It is easy to check that μ^φ defines a measure on $(\Omega', \mathcal{B}')$. (The reader is warned not to confuse the measure μ^φ with the Borel measure μ_u generated by the right continuous BV function u.) One also has the formula.

Proposition 2.1. *If $f : \Omega' \to \mathbb{R}$ is a measurable function, then*

$$\int_{\Omega'} f\, d\mu^\varphi = \int_\Omega \big(f \circ \varphi\big) d\mu \tag{2.6}$$

whenever either side is defined.

The proof of (2.6) follows the familiar process of first proving it for characteristic functions $f = \mathbb{1}_E$ with $E \in \mathcal{B}'$, by using $\mathbb{1}_E \circ \varphi = \mathbb{1}_{\varphi^{-1}(E)}$ and (2.5), then for simple functions and finally for integrable functions; cf. [11, p. 287]. In probability theory, when μ is a probability measure and $\varphi : \Omega \to \mathbb{R}$ is a Borel-measurable real-valued function, the image measure μ^φ is called the distribution of the random variable φ.

For u a right continuous function of bounded variation, let $L^1(du)$ denote the integrable functions with respect to the (signed) vector measure du. For instance, all the bounded, Borel measurable functions belong to $L^1(du)$. Proposition 2.1 provides certain change of variable formulas for Borel–Stieltjes integrals that are used extensively in the sequel.

THEOREM 2.2. *Let $u : [a,b] \to \mathbb{R}^N$ be a right continuous function of bounded variation, and let $X : [0,1] \to [a,b]$ be a continuous increasing (not necessarily strictly increasing) change of variables with $X(0) = a$, $X(1) = b$.*

(a) *If X^{-1} denotes the left-continuous inverse of X, then, for $f \in L^1\big(d(u \circ X)\big)$, we have*

$$\int_{[0,1]} f(s)\, d\big(u \circ X\big)(s) = \int_{[a,b]} f \circ X^{-1}(x)\, du(x). \tag{2.7}$$

(b) *For any function $g \in L^1\big(du\big)$, we have*

$$\int_{[0,1]} \big(g \circ X\big)(s)\, d\big(u \circ X\big)(s) = \int_{[a,b]} g(x)\, du(x). \tag{2.8}$$

Formula (2.8) when du is the Lebesgue measure is stated as an exercise in Folland [11, p. 103]. It is easy to construct examples showing that (2.8) fails if the hypothesis "X continuous" is replaced by "X right continuous."

Proof. We first establish (2.7) and (2.8) under the hypotheses

$$\begin{aligned} &u : [a,b] \to \mathbb{R} \quad \text{increasing and right continuous,}\\ &f : [0,1] \to [0,\infty] \quad \text{Borel measurable,}\\ &g : [a,b] \to [0,\infty] \quad \text{Borel measurable.} \end{aligned} \tag{2.9}$$

Since X is increasing, the inverse of X is a multivalued increasing map. We select the single-valued left-continuous inverse $\varphi = X^{-1}$ of the map X. Note that $X \circ \varphi = id$, but in general $\varphi \circ X \neq id$. The function $\varphi : [a,b] \to [0,1]$ is single valued, increasing, and satisfies

$$\varphi^{-1}\big((s,\tau]\big) = (X(s), X(\tau)] \quad \text{for } s, \tau \in [0,1].$$

Since the half-open intervals generate the Borel σ-algebra, φ is a $(\mathcal{B}_{[a,b]}, \mathcal{B}_{[0,1]})$-measurable map, that is, a Borel measurable map. Also, $f \circ \varphi$ is Borel measurable as well.

Let μ_u be the Borel measure generated by u, and let μ_u^φ be the image measure of μ_u under φ. Then

$$\mu_u^\varphi\big((s,\tau]\big) = \mu_u\big(\varphi^{-1}\big((s,\tau]\big)\big) = \mu_u\big((X(s), X(\tau)]\big) = \mu_{u\circ X}\big((s,\tau]\big)\,.$$

Since μ_u^φ and $\mu_{u\circ X}$ agree on the half-open intervals, the extension theorems for premeasures (e.g., [11, Thms. 1.14 and 1.16]) imply $\mu_u^\varphi = \mu_{u\circ X}$ on the Borel sets $\mathcal{B}_{[0,1]}$. Formula (2.7) is then a consequence of Proposition 2.1. In turn, (2.8) follows from (2.7), upon setting $f = g \circ X$ and using the identity $X \circ \varphi = id$.

Once (2.7) and (2.8) are established under (2.9), they are extended to hold under the hypotheses of Theorem 2.2. Consider, for instance, (2.7). It is first extended to hold for Borel measurable functions $f : [0,1] \to \mathbb{R}$ that are integrable with respect to $d(u \circ X)$, by using the decomposition $f = f^+ - f^-$, with f^+, $f^- \in L^1\big(d(u \circ X)\big)$. Next, if $u : [a,b] \to \mathbb{R}$ is a function of bounded variation, it can be decomposed in the form $u = u_1 - u_2$, with u_1, u_2 increasing and thus du_1, du_2 positive measures. Using the induced decomposition $d(u \circ X) = d(u_1 \circ X) - d(u_2 \circ X)$ of the signed measure $d(u \circ X)$ into a difference of positive measures, we can extend (2.7) to hold in this case also. Finally, the extension to the vector-valued case is trivial. ∎

Theorem 2.2 also yields a simple proof of the chain rule for Lipschitz functions in the one-dimensional context (see Marcus and Mizel [21], Boccardo and Murat [2]).

COROLLARY 2.3. *Suppose that $u : [a,b] \to \mathbb{R}^N$ is absolutely continuous and $X : [0,1] \to [a,b]$ is increasing, continuous, and onto. Then*

$$d\big(u \circ X\big) = \big(u' \circ X\big)\, dX. \tag{2.10}$$

If X is absolutely continuous, then

$$\frac{d}{ds}\big(u \circ X\big)(s) = u'\big(X(s)\big)\,\frac{dX}{ds}(s) \qquad \text{for almost everywhere (a.e.)} \quad s \in [0,1]. \tag{2.11}$$

Proof. We will show that

$$\int_0^s d(u \circ X) = \int_0^s u'(X(s))\, dX(s) \quad \text{ for } s \in [0,1].$$

Fix $s \in [0,1]$ and let $y = X(s)$ and $\bar{s} = \inf\{s \in [0,1] : X(s) > y\}$. Then $X(\tau) = y$ on the interval $[s, \bar{s}]$, and (2.8) in Theorem 2.2 implies

$$\begin{aligned}\int_0^s d(u \circ X) &= \int_0^{\bar{s}} d(u \circ X) = \int_{[a,y]} du(x) = \int_{[a,y]} u'(x)dx \\ &= \int_0^{\bar{s}} u'(X(s))dX(s) = \int_0^s u'(X(s))dX(s).\end{aligned}$$

Hence, (2.10) follows.

If X is absolutely continuous, then $u \circ X$ is also absolutely continuous and (2.11) follows from (2.10). ∎

Let $BV[a,b]$ be the set of all functions $u : [a,b] \to \mathbb{R}^N$ of bounded variation. The space $BV[a,b]$ can be identified to the space of (equivalence classes of) functions u in $L^1(a,b)$ whose distributional derivative, du/dx, is a finite, signed Borel measure. To see that, let $u \in BV[a,b]$ and let $\bar{u}$ denote a right continuous BV function such that $u = \bar{u}$ a.e. (the function $\bar{u}$ is uniquely determined by the equivalence class of u). By the Riesz representation theorem, the signed Borel measure $d\bar{u}$, generated by $\bar{u}$, can be identified with a bounded linear functional $\nu_{\bar{u}}$ on $\mathcal{C}[a,b]$,

$$\langle \nu_{\bar{u}}, \theta \rangle = \int_{[a,b]} \theta(x) d\bar{u}(x) \quad \text{ for } \theta \in \mathcal{C}[a,b]. \tag{2.12}$$

Then (2.4) implies that, for $\varphi \in \mathcal{C}_c^1(a,b)$,

$$\langle \nu_{\bar{u}}, \varphi \rangle = \int_{(a,b)} \varphi(x) d\bar{u}(x) = - \int_{(a,b)} \varphi'(x) u(x) dx\,, \tag{2.13}$$

i.e., the distributional derivative of u satisfies $du/dx = \nu_{\bar{u}}$. Moreover,

$$|\nu_{\bar{u}}|\big([a,b]\big) = TV_{[a,b]}(\bar{u})\,. \tag{2.14}$$

We note that if another representative is used on the right of (2.14), then equality is in general replaced by a strict inequality. The space $BV[a,b]$, when equipped with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |\nu_{\bar{u}}|\big([a,b]\big),$$

becomes a Banach space. For functions of one variable, it is customary to use the equivalent norm

$$\|u\|_{BV} = \|u\|_{L^\infty} + |\nu_{\bar{u}}|\big([a,b]\big).$$

We refer to Folland [11] and Volpert [31] for further information on the theory of BV functions.

2.2. Reparametrizations and distance of graphs. We present first the notion of uniform graph convergence [3, 10], which emerges when continuous paths are studied from the viewpoint of identifying two paths if their ranges coincide. In $\mathcal{C}[0,1]$, the space of continuous paths $V : [0,1] \to \mathbb{R}^M$, an equivalence relation is introduced.

DEFINITION 2.4. *We say that V_1 and V_2 are equivalent, $V_1 \sim V_2$, if and only if there exist two continuous, increasing (but not necessarily strictly increasing) and surjective maps $\gamma_1, \gamma_2 : [0,1] \to [0,1]$ such that $V_1 \circ \gamma_1 = V_2 \circ \gamma_2$.*

The following lemma is proved in [3, Lemma 1].

LEMMA 2.5. *Let $V_1, V_2 \in \mathcal{C}[0,1]$. Given two continuous, increasing, and surjective maps $\gamma_1, \gamma_2 : [0,1] \to [0,1]$ there exist two increasing, surjective maps $\alpha_1, \alpha_2 : [0,1] \to [0,1]$, Lipschitz continuous with Lipschitz constant 3, such that*

$$\max_{[0,1]} \big|V_1 \circ \alpha_1 - V_2 \circ \alpha_2\big| = \max_{[0,1]} \big|V_1 \circ \gamma_1 - V_2 \circ \gamma_2\big|.$$

Therefore, $V_1 \sim V_2$ if and only if there exist two Lipschitz continuous, increasing, and surjective maps $\alpha_1, \alpha_2 : [0,1] \to [0,1]$ such that $V_1 \circ \alpha_1 = V_2 \circ \alpha_2$.

If V is continuous and of bounded variation (and $TV_{[0,1]}(V) \neq 0$), then $V^c : [0,1] \to \mathbb{R}^M$, the *canonical parametrization* of V, is defined by

$$V^c(\tau) = V(s),\ \ \tau = \frac{1}{L}\,T_V(s), \qquad \text{where } L := TV_{[0,1]}(V), \tag{2.15}$$

the total variation function T_V being defined by (2.1). It is easy to check that V^c is well defined and, for $\tau_1 < \tau_2$,

$$|V^c(\tau_2) - V^c(\tau_1)| = |V(s_2) - V(s_1)| \le T_V(s_2) - T_V(s_1) = L(\tau_2 - \tau_1). \tag{2.16}$$

Hence, $V = V^c \circ \gamma$ where V^c is a Lipschitz continuous path and $\gamma = (1/L)T_V$ is continuous. The equivalence relation separates $\mathcal{C}[0,1]$ into equivalence classes that satisfy the following properties:

(1) If $V_1 \sim V_2$ and V_1 is of bounded variation, then V_2 is of bounded variation.

(2) If V is of bounded variation, then a Lipschitz continuous representative of the class can be selected, V^c.

(3) If V_1, V_2 are of bounded variation, then $V_1 \sim V_2$ if and only if $V_1^c = V_2^c$.

Statements (1) and (2) are clear. To show (3), suppose that $V_1 \sim V_2$ are of bounded variation and introduce the canonical parametrizations $V_\imath = V_\imath^c \circ \gamma_\imath$, where $\gamma_\imath = (1/L_\imath) T_{V_\imath}$ for $\imath = 1, 2$. Let $\alpha_1, \alpha_2 : [0,1] \to [0,1]$ be Lipschitz continuous, increasing, surjective maps such that $V_1 \circ \alpha_1 = V_2 \circ \alpha_2$. Then $T_{V_1} \circ \alpha_1 = T_{V_2} \circ \alpha_2$, $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$, and thus $V_1^c = V_2^c$.

On the space of continuous paths, we define a distance function:

$$\operatorname{dist}(V_1, V_2) := \inf_{\gamma_1, \gamma_2} \max_{s \in [0,1]} |(V_1 \circ \gamma_1)(s) - (V_2 \circ \gamma_2)(s)|, \tag{2.17}$$

where the infimum is taken over all continuous, increasing, and surjective maps $\gamma_1, \gamma_2 : [0,1] \to [0,1]$. Bressan and Rampazzo [3] introduce the distance and show that it defines a pseudometric,

$$\begin{aligned}
&\operatorname{dist}(V_1, V_2) = \operatorname{dist}(V_2, V_1),\\
&\operatorname{dist}(V, V) = 0,\\
&\operatorname{dist}(V_1, V_3) \le \operatorname{dist}(V_1, V_2) + \operatorname{dist}(V_2, V_3),
\end{aligned}$$

and that, by virtue of Lemma 2.5, the infimum in (2.15) is attained on two Lipschitz continuous paths α_1, α_2, so that the distance can be computed by

$$\operatorname{dist}(V_1, V_2) = \max_{s \in [0,1]} |(V_1 \circ \alpha_1)(s) - (V_2 \circ \alpha_2)(s)|.$$

In particular, that implies $\operatorname{dist}(V_1, V_2) = 0$ if and only if $V_1 \sim V_2$ and, thus, if the distance is viewed on the quotient space $\mathcal{X} = [\mathcal{C}([0,1]; \mathbb{R}^M)/\sim]$, it induces a metric. (Working with equivalence classes has the disadvantage of being cumbersome and identifying otherwise different functions; we will avoid doing that directly, but it is instructive to keep the structure in mind.) The associated convergence is called uniform graph convergence and is denoted by $V_n \overset{\mathrm{d}}{\to} V$: $\{V_n\}$ converges in graph to V if $\operatorname{dist}(V_n, V) \to 0$. Equivalently, $V_n \overset{\mathrm{d}}{\to} V$ if there exist two Lipschitz continuous, increasing, surjective maps $\alpha_n, \alpha : [0,1] \to [0,1]$ such that

$$\operatorname{dist}(V_n, V) = \max_{[0,1]} |V_n \circ \alpha_n - V \circ \alpha| \to 0 \quad \text{as } n \to \infty.$$

Finally we state a compactness result in Proposition 2.6.

PROPOSITION 2.6. *Let $\{V_n\}$ be a sequence of continuous functions on $[0,1]$ that are of uniformly bounded total variation. There exists a subsequence $\{V_{n_k}\}$ and a Lipschitz continuous representative $V^c : [0,1] \to \mathbb{R}^M$ such that $V_{n_k} \overset{d}{\to} V^c$.*

Proof. Let V_n^c be the canonical representatives of V_n, say, $V_n = V_n^c \circ \gamma_n$. By (2.15)–(2.16), V_n^c are uniformly Lipschitz continuous, with Lipschitz constant equal to the uniform variation bound of the sequence V_n. Since $V_n \sim V_n^c$, Lemma 2.5 implies there exist sequences α_n, β_n of uniformly Lipschitz continuous parametrizations such that $V_n \circ \alpha_n = V_n^c \circ \beta_n$. By the Ascoli–Arzela theorem there exist subsequences $V_{n_k}^c$, $\alpha_{n_k}, \beta_{n_k}$, and Lipschitz continuous functions $V^c : [0,1] \to \mathbb{R}^M$ and $\alpha, \beta : [0,1] \to [0,1]$ so that $V_{n_k}^c \to V^c$, $\alpha_{n_k} \to \alpha$, and $\beta_{n_k} \to \beta$ uniformly on $[0,1]$. Then

$$\begin{aligned}
V_{n_k} \circ \alpha_{n_k} &= V_{n_k}^c \circ \beta_{n_k}\\
&= (V_{n_k}^c \circ \beta_{n_k} - V_{n_k}^c \circ \beta) + V_{n_k}^c \circ \beta \to V^c \circ \beta
\end{aligned}$$

uniformly on $[0,1]$ and, thus, $V_{n_k} \overset{\mathrm{d}}{\to} V^c$. $\square$

2.3. Nonconservative products for continuous BV functions. The concepts of canonical parametrization and distance of continuous paths have implications when applied to graphs of continuous functions of bounded variation.

Let $u : [a,b] \to \mathbb{R}^N$ be a continuous function of bounded variation. The graph of u,

$$gr(u) := \big\{(x, u(x)) : x \in [a,b]\big\}, \tag{2.18}$$

is a continuous curve in $\mathbb{R} \times \mathbb{R}^N$. We introduce a canonical representative in the spirit of (2.15) (cf. [10]). Let $\sigma : [a,b] \to [0,1]$ be defined by

$$\sigma(x) := \frac{1}{L}\big(x - a + T_u(x)\big), \qquad \text{where } \; L := b - a + TV_{[a,b]}(u) > 0. \tag{2.19}$$

Then σ is strictly increasing, continuous, and surjective and satisfies $\sigma(a) = 0 < \sigma(x) < 1 = \sigma(b)$ for $x \in (a,b)$. The inverse of σ is a function $X : [0,1] \to [a,b]$, which is strictly increasing, continuous, and surjective. If we set $U := u \circ X$, the function $(X,U) : [0,1] \to [a,b] \times \mathbb{R}^N$ is a representative of the graph of u. Further, if $s_1 < s_2$ in $[0,1]$ and y_1, y_2 their respective images under X, $\sigma(y_1) = s_1$ and $\sigma(y_2) = s_2$, then

$$\begin{aligned} X(s_2) - X(s_1) = y_2 - y_1 \le L(\sigma(y_2) - \sigma(y_1)) = L(s_2 - s_1), \\ |U(s_2) - U(s_1)| = |u(y_2) - u(y_1)| \le T_u(y_2) - T_u(y_1) \le L(s_2 - s_1). \end{aligned} \tag{2.20}$$

Hence, (X,U) is Lipschitz continuous with Lipschitz constant L and will be referred to as the *arc-length parametrization* (or canonical representative) of the graph of $u \in \mathcal{C} \cap BV$.

The terminology "arc-length parametrization" is justified as follows: Since $T_u \circ X = T_{u\circ X} = T_U$, the parametrization (X,U) satisfies

$$s = \sigma(X(s)) = \frac{1}{L}\big(X(s) - a + T_U(s)\big) \tag{2.21}$$

for s in $[0,1]$. Therefore, (2.21) implies

$$\frac{dX}{ds} + \left|\frac{dU}{ds}\right| = L, \tag{2.22}$$

which means that the tangent vector to the curve $(X(s), U(s))$ has constant length equal to L. Strictly speaking, the arc-length parametrization corresponds to $L = 1$ in (2.22). This can be attained by stretching the interval $[0,1]$, but we avoid that here.

The graph of a continuous BV function u may be represented by several continuous, increasing, and surjective parametrizations $(Y,V) : [0,1] \to [a,b] \times \mathbb{R}^N$ with Y increasing. The representative can always be chosen to be a Lipschitz continuous path (X,U) with X strictly increasing. The distance between two graphs represented by (Y,V) and $(\bar{Y},\bar{V})$ is defined by $\operatorname{dist}\big((Y,V),(\bar{Y},\bar{V})\big)$ as in (2.17). The notion of distance and the equivalence relation $\sim$ provide a suitable tool for factoring representatives of the same graph (viewed as a geometric object). In what follows, we use the notation $(Y,V) \sim gr(u)$ to denote the general continuous representative (Y,V) of the graph of u and retain the notation (X,U) for the arc-length parametrization or for the associated notion of generalized graph defined in section 3.1.

The arc-length parametrization (X,U) may be used to express the Borel measure du generated by a continuous function of bounded variation u. Using Theorem 2.2, for the change of variable $x = X(s)$, we obtain

$$\int_{[a,b]} \theta(x)\, du(x) = \int_0^1 \big(\theta \circ X\big)(s) \frac{dU}{ds}\, ds \quad \text{for } \theta \in \mathcal{C}[a,b]. \tag{2.23}$$

The left side in (2.23) is interpreted as a Borel–Stieltjes integral, while the right side is a Lebesgue integral; the formula is useful for theoretical computations involving the measure du. If (Y,V) is an equivalent continuous representative of $gr(u)$, $(Y,V) \sim (X,U)$, repeated use of Theorem 2.2 implies

$$\int_{[a,b]} \theta(x)\, du(x) = \int_0^1 \big(\theta \circ X\big)(s)\, dU(s) = \int_0^1 \big(\theta \circ Y\big)(s)\, dV(s).$$

That is, the Borel measure du depends on $gr(u)$ but not on the particular representative.

We turn now to the definition of nonconservative products for *continuous* functions of bounded variation. A natural way of defining $\mu = g(u)\frac{du}{dx}$ is as a Borel measure, via (1.2). The definition is invariant under reparametrizations of $gr(u)$ and reads

$$\Big\langle g(u)\frac{du}{dx},\, \theta \Big\rangle = \int_0^1 \big(\theta \circ Y\big)(s) g(V(s))\, dV(s) = \int_0^1 \big(\theta \circ X\big)(s) g(U(s)) \frac{dU}{ds}\, ds,$$

where (X,U) is the arc-length parametrization and $(Y,V) \sim gr(u)$ stands for a general representative of the graph of u. This definition is consistent with the one proposed in section 4 for discontinuous BV functions.

3. Generalized graphs.

3.1. Generalized graphs of BV functions. The graph of a general function $u : [a,b] \to \mathbb{R}^N$ of bounded variation has jumps at the points of discontinuity of u. The notion of generalized graph (or graph completion), introduced by Bressan and Rampazzo [3], is an attempt to fill in the jumps by extending the idea of arc-length (or canonical) parametrization.

DEFINITION 3.1. *A generalized graph of u is a map $(X,U) : [0,1] \to [a,b] \times \mathbb{R}^N$ such that X, U are Lipschitz continuous and satisfy the following conditions:*

(1) $(X(0),U(0)) = (a,u(a))$, $(X(1),U(1)) = (b,u(b))$.

(2) *X is increasing: $s_1 < s_2$ implies $X(s_1) \le X(s_2)$.*

(3) *Given $y \in [a,b]$, there exists $s \in [0,1]$ such that $X(s) = y$, $U(s) = u(y)$.*

The range of (X,U) is a compact, connected set containing the graph of u. Let $\sigma = X^{-1}$ be the set theoretic inverse of X; then $\sigma : [a,b] \to [0,1]$ is a strictly increasing, *multivalued* map. The set $\mathcal{C}_\sigma$ of points of continuity of σ (that is, the point where σ is single valued) is dense in $[a,b]$. The set $\mathcal{S}_\sigma$ of points of discontinuity of σ (that is, the points where σ is truly multivalued) is countable and serves as a counter of the jumps and possible loops attached to the graph of u. In this paper, a point $x \in \mathcal{S}_\sigma$ is called a *point of jump* if $u(x-) \ne u(x+)$ and a *loop* if $u(x-) = u(x+)$.

The domain and range of σ admit the decompositions $[a,b] = \mathcal{C}_\sigma \cup \mathcal{S}_\sigma$ and

$$[0,1] = \sigma(\mathcal{C}_\sigma) \cup \sigma(\mathcal{S}_\sigma) = \Big(\bigcup_{y \in \mathcal{C}_\sigma} \{\sigma(y)\} \Big) \cup \Big(\bigcup_{y \in \mathcal{S}_\sigma} [\sigma(y-), \sigma(y+)] \Big), \tag{3.1}$$

respectively. The function u is recovered by the formula

$$u(y) = U(\sigma(y)) \quad \text{for } y \in \mathcal{C}_\sigma. \tag{3.2}$$

The following theorem indicates that the notion of generalized graph captures the limiting graphs selected by pointwise convergent sequences $\{u_n\}$ of *continuous* functions that are stable in $BV[a,b]$. Part (a) of the theorem below provides an extension (and an alternative proof) of the classical Helly selection principle.

THEOREM 3.2. (a) *Let* $\{u_n\}$ *be a sequence of continuous functions* $u_n : [a,b] \to \mathbb{R}^N$ *satisfying the uniform bounds* (1.3) *and let* (X_n, U_n) *be the arc-length parametrizations of* $gr(u_n)$. *There exists a subsequence* $\{u_{n_k}\}$, *a function of bounded variation* $u : [a,b] \to \mathbb{R}^N$, *and an associated generalized graph* (X, U) *such that*

(1) $(X_{n_k}, U_{n_k}) \xrightarrow{d} (X, U)$;

(2) $\sigma_{n_k}(y) \to \sigma(y)$, $u_{n_k}(y) \to u(y)$ *for all* $y \in \mathcal{C}_\sigma$ *and a.e. in* $[a,b]$,

where $\sigma_n = X_n^{-1}$ *and* $\sigma = X^{-1}$ *are the set theoretic inverses of* X_n *and* X, *respectively.*

(b) *Conversely, given a generalized graph* (X,U) *associated with a BV function* u, *there exists a sequence* $\{u_n\}$ *of Lipschitz continuous functions such that*

(1) $\{u_n\}$ *is uniformly bounded in BV*,

(2) $(Y_n, V_n) \xrightarrow{d} (X, U)$ *for any representative* $(Y_n, V_n) \sim gr(u_n)$,

(3) $u_n(y) \to u(y)$ *for* $y \in \mathcal{C}_\sigma$ *and a.e. in* $[a,b]$.

The proof is based on the following lemma.

LEMMA 3.3. *Suppose that* $(X_n, U_n) : [0,1] \to [a,b] \times \mathbb{R}^N$ *satisfy the following conditions:*

(1) X_n *is strictly increasing and surjective,*

(2) (X_n, U_n) *are uniformly Lipschitz continuous,*

(3) $(X_n, U_n) \to (X, U)$ *uniformly on* $[0,1]$.

Let $\sigma_n = X_n^{-1}$, $u_n = U_n \circ X_n^{-1}$. *Then* (X_n, U_n) *is a Lipschitz continuous representative of* $gr(u_n)$ *and*

$$\operatorname{dist}\big((Y_n, V_n), (X, U)\big) \to 0 \quad \text{for any representative } (Y_n, V_n) \sim gr(u_n),$$
$$\sigma_n(y) \to \sigma(y), \ \ u_n(y) \to u(y) \quad \text{for all } y \in \mathcal{C}_\sigma.$$

Proof. Since X_n is strictly increasing, the functions $\sigma_n = X_n^{-1} : [a,b] \to [0,1]$ and $u_n = U_n \circ X_n^{-1}$ are well defined and continuous. The couple (X_n, U_n) is a representative of the graph of u_n.

Fix $y \in \mathcal{C}_\sigma$ and let $s = \sigma(y)$, $s_n = \sigma_n(y)$. We can write the chain of identities

$$\begin{aligned} s_n - s = \sigma_n(y) - \sigma(y) &= \sigma\Big(X\big(\sigma_n(y)\big)\Big) - \sigma\Big(X_n\big(\sigma_n(y)\big)\Big) \\ &= \sigma\Big(X(s_n)\Big) - \sigma\Big(X_n(s_n)\Big). \end{aligned}$$

Since $X_n \to X$ uniformly and $X_n(s_n) = y \in \mathcal{C}_\sigma$, we deduce $s_n \to s$.

Next, assumption (2) implies

$$|U_n(s_n) - U_n(s)| \le \operatorname{Lip}(U_n)\, |s_n - s| \to 0.$$

Hence, $u_n(y) = U_n(s_n) \to U(s) = u(y)$.

Finally, if (Y_n, V_n) is any continuous representative of $gr(u_n)$, then

$$\operatorname{dist}\big((Y_n, V_n), (X, U)\big) \le \operatorname{dist}\big((X_n, U_n), (X, U)\big) \to 0 \quad \text{as } n \to \infty.$$

This completes the proof. □

Proof of Theorem 3.2. (a) The sequence $\{u_n\}$ consists of continuous functions. Let (X_n, U_n) be the arc-length parametrizations of $gr(u_n)$, defined by inverting

$$\sigma_n(x) := \frac{1}{L_n}\big(x - a + T_{u_n}(x)\big), \qquad L_n := b - a + T_{u_n}(b),$$

and setting $X_n := {\sigma_n}^{-1}$, $U_n := u_n \circ X_n$. In view of (2.22) and (1.3), (X_n, U_n) are uniformly Lipschitz continuous. There exists a subsequence (X_{n_k}, U_{n_k}) and a Lipschitz continuous function $(X,U) : [0,1] \to [a,b] \times \mathbb{R}^N$ such that

$$X_{n_k} \to X, \;\; U_{n_k} \to U \qquad \text{uniformly on } \; [0,1]. \tag{3.3}$$

Hence, $\operatorname{dist}((X_{n_k}, U_{n_k}), (X,U)) \to 0$ as $k \to \infty$, and (3.3), in conjunction with Lemma 3.3, yields the conclusion of part (a).

(b) Given a generalized graph (X,U), let (X_n, U_n) be defined by

$$X_n := \left(1 - \frac{1}{n}\right) X + \frac{1}{n}\big(a + (b-a)s\big), \qquad U_n := U.$$

Then $X_n : [0,1] \to [a,b]$ is strictly increasing, Lipschitz continuous, and surjective; $\{(X_n, U_n)\}$ are uniformly Lipschitz continuous, while $(X_n, U_n) \to (X,U)$ uniformly on $[0,1]$. The functions u_n, defined by $u_n = U_n \circ X_n^{-1}$, are Lipschitz continuous and satisfy

$$\begin{aligned} \sup_{[a,b]} |u_n| &= \sup_{[0,1]} |U|, \\ TV_{[a,b]}(u_n) &= TV_{[0,1]}(U) \leq Lip(U). \end{aligned}$$

The conclusion of part (b) now follows from Lemma 3.3. $\square$

It is instructive to place the above concepts in a functional analysis framework. Let

$$E = \big\{ (Y,V) \in \mathcal{C}\big([0,1]; [a,b] \times \mathbb{R}^N\big) \;\; : Y(0) = a, \, Y(1) = b \big\} \tag{3.4}$$

and $\mathcal{X} := (E/\sim)$ be the quotient space of E over the equivalence relation $\sim$ introduced in Definition 2.4. The elements of $\mathcal{X}$ are equivalence classes of functions: (Y_1, V_1), (Y_2, V_2) are in the same equivalence class if and only if $(Y_1, V_1) \circ \alpha = (Y_2, V_2) \circ \beta$ for some Lipschitz and increasing reparametrizations of $[0,1]$; that is, the curves determined by the functions (Y_1, V_1) and (Y_2, V_2) coincide. The elements of $\mathcal{X}$ can thus be visualized as geometric curves in $[a,b] \times \mathbb{R}^N$ with $Y(0) = a$, $Y(1) = b$.

If $(Y,V) \in E \cap BV$, one can select, using (2.19), a Lipschitz continuous representative of the equivalence class $[(Y,V)]$. This representative is denoted here by (X,U) and is characteristic to the class. The reason is that, for (Y,V) and $(\bar{Y}, \bar{V})$ of bounded variation, $(Y,V) \sim (\bar{Y}, \bar{V})$ if and only if the corresponding canonical representatives of $[(Y,V)]$ and $[(\bar{Y}, \bar{V})]$ are identical, $(X,U) = (\bar{X}, \bar{U})$. We emphasize that we can talk about the canonical representative only for $\mathcal{C} \cap BV$ curves. (Recall that if one representative of the equivalence class is of bounded variation, any representative is of bounded variation.)

When $\mathcal{X}$ is equipped with the pseudometric $\operatorname{dist}((Y,V), (\bar{Y}, \bar{V}))$, defined in (2.17), it becomes a metric space. Consider now the sets

$$\begin{aligned} &\mathcal{F} = \big\{ [(Y,V)] \in \mathcal{X} \;:\; (Y,V) \text{ is of bounded variation and } X \text{ is strictly increasing} \big\}, \\ &\mathcal{G} = \big\{ [(Y,V)] \in \mathcal{X} \;:\; (Y,V) \text{ is of bounded variation and } X \text{ is increasing} \big\}, \end{aligned} \tag{3.5}$$

where (X,U) always refers to the canonical representative of $[(Y,V)]$. Section 2.3 indicates that $\mathcal{F}$ can be identified with the set of continuous functions of bounded variation, $(\mathcal{C} \cap BV)\big([a,b]; \mathbb{R}^N\big)$. The elements of $\mathcal{F}$ are viewed as the graphs of the functions

u, with the canonical representative coinciding with the arc-length parametrization of $gr(u)$ in (2.19). The set $\mathcal{G}$ is the closure of $\mathcal{F}$ in the metric induced by the distance function (2.17) and may itself be viewed as a complete metric space. The canonical representative of each equivalence class element of $\mathcal{G}$ is a generalized graph in the sense of Definition 3.1. Henceforth, elements of $\mathcal{G}$ are denoted by $gr(X, U)$ and are visualized as the geometric graphs generated by (X, U). We remark that elements of $\mathcal{G}$ are not in correspondence with the space of BV functions, but rather $\mathcal{G}$ consists of all possible limit points of $\mathcal{F}$ in the distance metric.

3.2. Representation of weak-$\star$ limits. Consider a sequence $\{u_n\}$ of *continuous* functions satisfying the uniform bounds (1.3). The sequence $\{g(u_n)du_n\}$ may have multiple limit points in the weak-$\star$ topology of $\mathcal{M}[a, b]$ (cf. Example 1.1). We now characterize such limits for any continuous g in the following representation theorem.

THEOREM 3.4. (a) *Let $\{u_n\}$ be a sequence of continuous functions satisfying the uniform bounds* (1.3). *There exists a subsequence $\{u_{n_k}\}$ and a generalized graph (X, U) such that, for any continuous function $g = g(\lambda)$, we have*

$$\int_{[a,b]} \theta(x) g(u_{n_k}(x))\, du_{n_k}(x) \to \langle \mu(g), \theta\rangle \quad \textit{for} \quad \theta \in \mathcal{C}[a, b], \tag{3.6}$$

where $\mu : \mathcal{C}_0(\mathbb{R}^N) \to \mathcal{M}[a, b]$ is defined by

$$\langle \mu(g), \theta\rangle = \int_0^1 \theta(X(s)) g(U(s))\, dU(s). \tag{3.7}$$

(b) *Conversely, given a generalized graph (X, U), let μ be defined by* (3.7). *There exists a sequence of Lipschitz functions $\{u_n\}$, uniformly bounded in BV, such that for any continuous g,*

$$g(u_n)du_n \rightharpoonup \mu(g) \quad \textit{weak-}\star \textit{ in } \mathcal{M}[a, b]. \tag{3.8}$$

Theorem 3.4 is based on a characterization of the weak-$\star$ limit points to the sequence of Radon measures $\{p_n\}$ defined in (1.9). The key ingredient is the following weak stability type of theorem.

THEOREM 3.5. *Let $\{u_n\}$ be a sequence of continuous functions $u_n : [a, b] \to \mathbb{R}^N$ satisfying the uniform bounds* (1.3), *and let (X_n, U_n) be the arc-length parametrization of $gr(u_n)$. If*

$$(X_n, U_n) \xrightarrow{d} (X, U) \tag{3.9}$$

to some generalized graph (X, U) associated with a BV function u, then

$$\int_{[a,b]} f(x, u_n(x))\, du_n(x) \to \int_0^1 f(X(s), U(s)) dU(s) \quad \textit{for } f \in \mathcal{C}_0([a, b] \times \mathbb{R}^N). \tag{3.10}$$

Proof. Let (X_n, U_n) be the arc-length parametrizations of $gr(u_n)$, and let (X, U) be a generalized graph of u. Hypothesis (3.9) implies that for some α_n and α, Lipschitz continuous reparametrizations of the interval $[0, 1]$, we have

$$\begin{aligned} Y_n &:= X_n \circ \alpha_n \to X \circ \alpha =: Y \qquad \text{uniformly on } [0, 1], \\ V_n &:= U_n \circ \alpha_n \to U \circ \alpha =: V \qquad \text{uniformly on } [0, 1]. \end{aligned}$$

By virtue of Theorem 2.2, we may express the integrals

$$\int_{[a,b]} f(x, u_n(x))\, du_n(x) = \int_0^1 f(X_n(s), U_n(s)) dU_n(s) = \int_0^1 f(Y_n(s), V_n(s)) dV_n(s),$$

$$\int_0^1 f(X(s), U(s)) dU(s) = \int_0^1 f(Y(s), V(s)) dV(s).$$

Fix $f \in \mathcal{C}_0\big([a,b] \times \mathbb{R}^N\big)$. Note that (Y_n, V_n) and (Y, V) are continuous and satisfy

$$\begin{aligned} &TV(V_n) = TV(U_n) \le C, \\ &(Y_n, V_n) \to (Y, V) \quad \text{uniformly on } [0,1]. \end{aligned} \tag{3.11}$$

It suffices to show that (3.11) implies

$$\int_0^1 f(Y_n(s), V_n(s)) dV_n(s) \to \int_0^1 f(Y(s), V(s)) dV(s). \tag{3.12}$$

Step 1. We first show that, if V_n, $V \ : [0,1] \to \mathbb{R}^N$ are functions of bounded variation (not necessarily continuous) such that

(1) $\|V_n - V\|_\infty \to 0$,

(2) $TV(V_n) \le C$,

then, for any $[\alpha, \beta] \subset [0,1]$ and $\varphi \in \mathcal{C}[\alpha, \beta]$, we have

$$\int_{[\alpha,\beta]} \varphi(s)\, dV_n(s) \to \int_{[\alpha,\beta]} \varphi(s)\, dV(s). \tag{3.13}$$

If $\psi \in \mathcal{C}^1[\alpha, \beta]$, then (2.4) implies

$$\begin{aligned} \int_{[\alpha,\beta]} \psi(s)\, dV_n(s) &= -\int_{[\alpha,\beta]} \psi'(s)\, V_n(s) ds + \psi(\beta) V_n(\beta+) - \psi(\alpha) V_n(\alpha-) \\ &\to -\int_{[\alpha,\beta]} \psi'(s)\, V(s) ds + \psi(\beta) V(\beta+) - \psi(\alpha) V(\alpha-) \\ &= \int_{[\alpha,\beta]} \psi(s)\, dV(s). \end{aligned} \tag{3.14}$$

Given $\varphi \in \mathcal{C}[\alpha, \beta]$, there exists for every $\varepsilon > 0$ a function $\psi \in \mathcal{C}^1[\alpha, \beta]$ such that $\|\psi - \varphi\|_\infty < \varepsilon$. The relation

$$\begin{aligned} &\left| \int_{[\alpha,\beta]} \varphi(s)\, dV_n(s) - \int_{[\alpha,\beta]} \varphi(s)\, dV(s) \right| \\ &\le \varepsilon\, TV_{[\alpha,\beta]}(V_n) + \left| \int_{[\alpha,\beta]} \psi(s)\, dV_n(s) - \int_{[\alpha,\beta]} \psi(s)\, dV(s) \right| + \varepsilon\, TV_{[\alpha,\beta]}(V), \end{aligned}$$

in conjunction with (3.14), yields (3.13).

Step 2. Step 1, in conjunction with (3.11), implies

$$\int_0^1 f(Y(s), V(s)) dV_n(s) \to \int_0^1 f(Y(s), V(s)) dV(s).$$

On the other hand, again by (3.11),

$$\left| \int_0^1 \big(f(Y_n, V_n) - f(Y, V)\big)\, dV_n \right| \le \big(\max |f(Y_n, V_n) - f(Y, V)|\big) \int_{[0,1]} |dV_n| \to 0,$$

as $n \to \infty$. Hence, (3.12) follows. ∎

Proof of Theorem 3.4. (a) Let $\{u_n\}$ be the sequence of continuous functions satisfying (1.3), and let (X_n, U_n) be the arc-length parametrizations of $gr(u_n)$; the latter are uniformly Lipschitz continuous. Let $\{p_n\}$ be the sequence of Radon measures defined in (1.9). The sequence $\{p_n\}$ is bounded, $\|p_n\|_{\mathcal{M}} \le C$, by the uniform BV bound of $\{u_n\}$.

Let p be a weak-$\star$ limit point of $\{p_n\}$. For a subsequence

$$\langle p_{n_k}, f(x,\lambda)\rangle = \int_{[a,b]} f(x, u_{n_k}) du_{n_k} \to \langle p, f(x,\lambda)\rangle \text{ for } f \in \mathcal{C}_0([a,b]\times\mathbb{R}^N).$$

Using part (a) of Theorem 3.2 and passing to a further subsequence $\{u_{n_k}\}$, if necessary, we may assume that there is a generalized graph (X, U), so that the arc-length parametrizations $(X_{n_k}, U_{n_k}) \xrightarrow{\mathrm{d}} (X, U)$. Theorem 3.5 implies

$$\langle p, f(x,\lambda)\rangle = \int_0^1 f(X(s), U(s)) dU(s).$$

Taking $f(x,\lambda) = \theta(x)g(\lambda)$ gives the desired result for $g \in \mathcal{C}_0(\mathbb{R}^N)$ and, due to the uniform sup-norm bound of $\{u_n\}$, for any continuous g.

(b) Given a generalized graph (X, U), let μ be defined by (3.7) and let $\{u_n\}$ be the sequence of Lipschitz functions constructed in the proof of part (b) of Theorem 3.2. Then $\{u_n\}$ are uniformly bounded in BV, $\{(X_n, U_n)\}$ are uniformly Lipschitz continuous, and $(X_n, U_n) \to (X, U)$ uniformly on $[0, 1]$. Theorem 3.5 for $f(x,\lambda) = \theta(x)g(\lambda)$ implies (3.8). ∎

4. Definition of nonconservative products.

4.1. Definition as a Radon measure. In view of Theorem 3.4, the definition of nonconservative products should be based on a given generalized graph $(X, U) : [0, 1] \to [a, b] \times \mathbb{R}^N$ of the function u of bounded variation. The generalized graph (X, U) determines a geometric object (the graph of u together with paths filling the jumps and possible attached loops), call it $gr(X, U)$. We define $g(u)\frac{du}{dx}$ relative to $gr(X, U)$, first as a Radon measure in this section, and then as a finite Borel measure via its distribution function in section 4.2.

DEFINITION 4.1. *Let $(Y, V) \sim gr(X, U)$ denote the general continuous representative of the graph determined by (X, U). Given a continuous map $g : \mathbb{R}^N \to \mathbb{R}^N$, define $\mu(g)$ by*

$$\begin{aligned}\langle \mu(g), \theta\rangle &= \int_0^1 \theta(Y(s))g(V(s))\, dV(s)\\ &= \int_0^1 \theta(X(s))g(U(s))\frac{dU}{ds}\, ds \quad \textit{for } \theta \in \mathcal{C}[a,b].\end{aligned}\tag{4.1}$$

Then $\mu(g) \in \mathcal{M}[a, b]$ is called the nonconservative product of $g(u)$ by $\frac{du}{dx}$ and is denoted by

$$\mu(g) = \left[g(u)\frac{du}{dx}\right]_{(X,U)}.\tag{4.2}$$

Remark 4.2. (a) We refer to Dal Maso, LeFloch, and Murat [10] for a slightly weaker definition of nonconservative products and to Raymond [27] for a definition that is equivalent. Comparisons of the various definitions are carried out in section 4.4. References [10, 27] also contain various weak stability results.

(b) Suppose that (Y, V), $(\bar{Y}, \bar{V})$ are two representatives of the same graph, that is, $(Y, V) \sim (\bar{Y}, \bar{V})$. Then $(Y, V) \circ \beta = (\bar{Y}, \bar{V}) \circ \alpha$ for some Lipschitz reparametrizations α, β of $[0, 1]$. Theorem 2.2 implies the nonconservative product remains invariant,

$$\int_0^1 \theta(Y(s)) g(V(s))\, dV(s) = \int_0^1 \theta(\bar{Y}(s)) g(\bar{V}(s))\, d\bar{V}(s). \tag{4.3}$$

The measure introduced in Definition 4.1 thus depends on the equivalence class determined by the generalized graph (X, U), i.e., on $gr(X, U)$ as a geometric object. When a Lipschitz representative, such as (X, U) itself is used, then $\langle \mu(g), \theta \rangle$ may be expressed via the last integral in (4.1).

(c) If μ is viewed as a map $\mu\ : \mathcal{C}_0(\mathbb{R}^N) \to \mathcal{M}[a, b]$, then μ is linear and bounded. The boundedness follows from the estimate

$$|\langle \mu(g), \theta \rangle| \leq \big(TV_{[0,1]}(V)\big) \sup_{|\lambda| \leq \max |U|} |g(\lambda)| \sup_{x \in [0,1]} |\theta(x)|, \tag{4.4}$$

which implies $\|\mu(g)\|_{\mathcal{M}} \leq (TV_{[0,1]}(V))\, \|g\|_{\mathcal{C}_0}$. ∎

We state next a weak stability theorem for nonconservative products.

Theorem 4.3. (i) *Let $\{(X_n, U_n)\}$ and (X, U) be generalized graphs. If*

(1) *$TV(U_n)$ is uniformly bounded,*

(2) *$(X_n, U_n) \xrightarrow{d} (X, U)$,*

then

$$\Big[g(u_n) \frac{du_n}{dx}\Big]_{(X_n, U_n)} \rightharpoonup \Big[g(u) \frac{du}{dx}\Big]_{(X,U)} \qquad \textit{weak-}\star \textit{ in } \mathcal{M}[a, b]. \tag{4.5}$$

(ii) *Let $\{u_n\}$ be a sequence of continuous functions satisfying* (1.3)*, let (X_n, U_n) be the arc-length parametrizations of $gr(u_n)$, and let (X, U) be a generalized graph. If $(X_n, U_n) \xrightarrow{d} (X, U)$, then*

$$g(u_n) du_n \rightharpoonup \Big[g(u) \frac{du}{dx}\Big]_{(X,U)} \qquad \textit{weak-}\star \textit{ in } \mathcal{M}[a, b]. \tag{4.6}$$

Proof. Define the graphs determined by (X_n, U_n) and (X, U), and let $(Y_n, V_n) \sim gr(X_n, U_n)$ and $(Y, U) \sim gr(X, U)$ be continuous representatives such that $(Y_n, V_n) \to (Y, V)$ uniformly on $[0, 1]$. Moreover, $TV(V_n) = TV(U_n) \leq C$. The result follows from (3.10) in Theorem 3.5, together with part (b) of Remark 4.2. ∎

4.2. Distribution functions. We discuss next the properties of nonconservative products when viewed as signed Borel measures defined via their distribution functions. Recall, for a generalized graph (X, U), the set theoretic inverse $\sigma = X^{-1}\ : [0, 1] \to [a, b]$ is a strictly increasing multivalued map.

Theorem and Definition 4.4. *Let $(Y, V) \sim gr(X, U)$ be a representative of the graph determined by (X, U). For $x \in [a, b]$ define*

$$\begin{aligned} F(x) &= \int_0^{Y^{-1}(x+)} g(V(s))\, dV(s) = \int_0^{X^{-1}(x+)} g(U(s)) \frac{dU}{ds}\, ds, \\ F(a-) &= 0. \end{aligned} \tag{4.7}$$

Then F is a right continuous BV function and generates a signed Borel measure μ, determined by

$$\mu((a,x]) = F(x) - F(a) \quad \text{for } x \in (a,b], \qquad \mu(\{a\}) = F(a). \tag{4.8}$$

Also μ coincides with the nonconservative product $[g(u)\frac{du}{dx}]_{(X,U)}$ in (4.1)–(4.2); that is,

$$\begin{aligned}\langle \mu, \theta\rangle &= \int_{[a,b]} \theta(x)\, dF(x) \\ &= \int_{[0,1]} \theta(Y(s))g(V(s))\, dV(s) = \int_0^1 \theta(X(s))g(U(s))\frac{dU}{ds}\, ds\end{aligned} \tag{4.9}$$

for any $\theta \in \mathcal{C}[a,b]$.

Proof. Consider $(Y,V) \sim gr(X,U)$, a general continuous representative of the graph determined by (X,U). We have the following conditions:

(i) $X\ : [0,1] \to [a,b]$ is Lipschitz continuous, increasing, and surjective with $X(0) = a$, $X(1) = b$, and $X^{-1}(X(s)) = s$ whenever $X(s) \in \mathcal{C}_{X^{-1}}$.

(ii) $Y\ : [0,1] \to [a,b]$ is continuous, increasing, and surjective with $Y(0) = a$, $Y(1) = b$, and $Y^{-1}(Y(s)) = s$ whenever $Y(s) \in \mathcal{C}_{Y^{-1}}$.

(iii) $(Y,V)\circ\beta = (X,U)\circ\alpha$ for some $\alpha,\beta\ : [0,1] \to [0,1]$ increasing, Lipschitz, and surjective reparametrizations.

Let $F\ : [a,b] \to \mathbb{R}^N$ be defined by

$$F(x) = \int_0^{Y^{-1}(x+)} g(V(s))\, dV(s), \quad F(a-) = 0.$$

Then F is a right continuous BV function and generates a signed Borel measure μ, through (4.8). Note that F satisfies

$$F(Y(s)) = \int_0^s g(V)dV \qquad \text{for } s \in Y^{-1}(\mathcal{C}_{Y^{-1}}). \tag{4.10}$$

Step 1. The definition of the distribution function F depends on the equivalence class of (X,U) but not on the specific representative.

It suffices to define F at points $x \in \mathcal{C}_{Y^{-1}}$ and to extend F so that it is right continuous. If $(Y,V) \sim (X,U)$ are two equivalent representatives of $gr(X,U)$, then $Y\circ\beta = X\circ\alpha$, $V\circ\beta = U\circ\alpha$, and $\mathcal{C}_{(X\circ\alpha)^{-1}} \subset \mathcal{C}_{X^{-1}}$, $\mathcal{C}_{(Y\circ\beta)^{-1}} \subset \mathcal{C}_{Y^{-1}}$. For $x \in \mathcal{C}_{(X\circ\alpha)^{-1}} = \mathcal{C}_{(Y\circ\beta)^{-1}}$, Theorem 2.2 implies

$$\begin{aligned}\big[F(x)\big]_{(X,U)} &:= \int_0^{X^{-1}(x)} g(U)dU = \int_0^{(X\circ\alpha)^{-1}(x)} g(U\circ\alpha)d(U\circ\alpha), \\ \big[F(x)\big]_{(Y,V)} &:= \int_0^{Y^{-1}(x)} g(V)dV = \int_0^{(Y\circ\beta)^{-1}(x)} g(V\circ\beta)d(V\circ\beta).\end{aligned}$$

Since such points are dense in $[a,b]$, any of these formulas generates the same distribution function $[F]_{(X,U)} = [F]_{(Y,V)}$ and we may use any representative for calculating F. This shows (4.7).

Step 2. For $\theta \in \mathcal{C}[a,b]$, we shall show that

$$\int_{[a,b]} \theta(x)\, dF(x) = \int_{[0,1]} \theta(X(s))g(U(s))\frac{dU}{ds} ds. \tag{4.11}$$

(Note that this formula is *not* a direct consequence of Theorem 2.2.)

Fix $\psi \in \mathcal{C}^1[a,b]$. Using (2.4), the change of variables $x = X(s)$, (4.7), (4.10), the property $\dot{X} = 0$ on each interval $[\sigma(y-), \sigma(y+)]$ with $y \in \mathcal{S}_\sigma$, and the chain rule for Lipschitz continuous functions, we obtain

$$\begin{aligned}
\int_{[a,b]} \psi(x)\, dF(x) =& \psi(b)F(b+) - \psi(a)F(a-) - \int_{[a,b]} \psi'(x)F(x)\, dx \\
=& \psi(b)F(b+) - \int_0^1 \psi'(X(s))F(X(s))\, \dot{X}(s)\, ds \\
=& \psi(b)F(b+) - \int_{\sigma(\mathcal{C}_\sigma)} \psi'(X(s))\dot{X}(s) \left(\int_0^s g(U)\frac{dU}{d\tau} d\tau \right) ds \\
& - \sum_{y\in\mathcal{S}_\sigma} \int_{\sigma(y-)}^{\sigma(y+)} \psi'(X(s))\dot{X}(s) \left(\int_0^{\sigma(X(s)+)} g(U)\frac{dU}{d\tau} d\tau \right) ds \\
=& \psi(X(1)) \int_0^1 g(U)\frac{dU}{d\tau} d\tau - \int_0^1 \frac{d}{ds}\big(\psi(X(s))\big) \left(\int_0^s g(U)\frac{dU}{d\tau} d\tau \right) ds;
\end{aligned}$$

thus

$$\int_{[a,b]} \psi(x)\, dF(x) = \int_0^1 \psi(X(s))g(U(s))\frac{dU}{ds}\, ds.$$

Since F is of bounded variation and U is Lipschitz continuous, a density argument yields (4.11). The proof of (4.9) follows from part (b) of Remark 4.2. ◻

Remark 4.5. In view of (4.7) and (4.8), the nonconservative product μ charges points $x \in \mathcal{S}_{X^{-1}}$ according to

$$\mu\big(\{x\}\big) = F(x+) - F(x-) = \int_{X^{-1}(x-)}^{X^{-1}(x+)} g(U(s))\, dU(s). \qquad ◻$$

We state and prove a version of the weak stability theorem by using distribution functions.

THEOREM 4.6. *Suppose $\{u_n\}$ is a sequence of continuous functions satisfying (1.3). Let (X_n, U_n) be the arc-length parametrizations of $gr(u_n)$, let (X,U) be a generalized graph, and define the distribution functions*

$$F_n(x) = \int_a^x g(u_n(y))\, du_n(y), \tag{4.12}$$

and $F(x)$ associated with (X,U) by (4.7). If $(X_n, U_n) \xrightarrow{d} (X,U)$, then

$$F_n(x) \to F(x) \qquad \text{a.e. in } (a,b), \tag{4.13}$$

while μ_n and μ, generated by F_n and F, respectively, satisfy $\mu_n \rightharpoonup \mu$ weak-$\star$ in $\mathcal{M}[a,b]$.

Proof. Let (X_n, U_n) be the arc-length parametrizations of $gr(u_n)$; (X_n, U_n) are uniformly Lipschitz. There exist reparametrizations of the interval $[0,1]$, α_n, and α that are uniformly Lipschitz continuous such that $(\bar{X}_n, \bar{U}_n) = (X_n, U_n)\circ\alpha_n$, $(\bar{X}, \bar{U}) = (X,U)\circ\alpha$ satisfy the following: $(\bar{X}_n, \bar{U}_n)$ are uniformly Lipschitz and $\bar{X}_n \to \bar{X}$, $\bar{U}_n \to \bar{U}$ uniformly on $[0,1]$.

Let $\mathcal{S}_{\bar{X}_n^{-1}}$ and $\mathcal{S}_{\bar{X}^{-1}}$ be the points of discontinuity of $\bar{X}_n^{-1}$ and $\bar{X}^{-1}$, respectively, and set $\mathcal{T} = (\bigcup_n \mathcal{S}_{\bar{X}_n^{-1}}) \cup \mathcal{S}_{\bar{X}^{-1}}$. Then $\mathcal{T}$ is countable, and an argument as in the proof of Lemma 3.3 shows

$$\bar{X}_n^{-1}(x) \to \bar{X}^{-1}(x) \quad \text{for } x \in [a,b] \setminus \mathcal{T}. \tag{4.14}$$

Theorem 2.2, for the change of variables $y = \bar{X}_n(s)$, gives

$$F_n(x) = \int_a^x g(u_n(y))\, du_n(y) = \int_0^{\bar{X}_n^{-1}(x)} g(\bar{U}_n(s)) d\bar{U}_n(s) \quad \text{for } x \in \mathcal{C}_{\bar{X}_n^{-1}}. \tag{4.15}$$

An argument, as in the proof of (3.12), shows that

$$\int_0^{\bar{X}^{-1}(x)} g(\bar{U}_n(s)) d\bar{U}_n(s) \to \int_0^{\bar{X}^{-1}(x)} g(\bar{U}(s)) d\bar{U}(s) \quad \text{for } x \in \mathcal{C}_{\bar{X}^{-1}}. \tag{4.16}$$

In turn, (4.14)–(4.16) and the fact that $\bar{U}_n$ are uniformly Lipschitz imply

$$F_n(x) \to F(x) \quad \text{for } x \in [a,b] \setminus \mathcal{T}. \tag{4.17}$$

The distribution functions F_n and F satisfy the following properties: $F_n(a) = 0$, $F(a-) = 0$,

$$F_n(b) = \int_0^1 g(\bar{U}_n(s)) d\bar{U}_n(s)\, ds \to \int_0^1 g(\bar{U}(s)) d\bar{U}(s) = F(b). \tag{4.18}$$

For any test function $\psi \in \mathcal{C}^1[a,b]$, the integration by parts formula (2.4), in conjunction with (4.17)–(4.18), yields

$$\begin{aligned} \int_{[a,b]} \psi(x) dF_n(x) &= -\int_{[a,b]} \psi'(x) F_n(x)\, dx + \psi(b) F_n(b) - \psi(a) F_n(a) \\ &\to -\int_{[a,b]} \psi'(x) F(x)\, dx + \psi(b) F(b) - \psi(a) F(a-); \end{aligned}$$

hence

$$\int_{[a,b]} \psi(x) dF_n(x) \quad \to \quad \int_{[a,b]} \psi(x) dF(x). \tag{4.19}$$

Since F_n are of uniformly bounded variation, (4.19) and a density argument show that $\mu_n \rightharpoonup \mu$ weak-$\star$ in $\mathcal{M}[a,b]$. $\square$

4.3. Generalized graphs and graphs of BV functions. In this section we examine the relation between a generalized graph (X,U) and the graph of the associated BV function u. First observe that Definition 3.1 directly implies the following proposition.

PROPOSITION 4.7. *Let (X,U) be a generalized graph and let $\sigma = X^{-1}$ be the set theoretic inverse of X. Then the following conditions hold:*

(i) *If $y \in \mathcal{C}_\sigma$, then $u(y) = U(\sigma(y))$.*

(ii) *If $y \in \mathcal{S}_\sigma$, then $X(\tau) = y$ and the function*

$$\phi_y(\tau) := U(\tau), \qquad \tau \in J_y := [\sigma(y-), \sigma(y+)],$$

determines a Lipschitz continuous curve that lies on the hyperplane $\{x = y\}$ *and connects* $(y, u(y-))$ *with* $(y, u(y+))$*.*

(1) *The Lipschitz path* $\phi_y : J_y \to \mathbb{R}^N$ *is either an arc when* $u(y-) \neq u(y+)$ *or a loop when* $u(y-) = u(y+)$*.*

(2) *The Lipschitz continuity of* U *implies*

$$\sum_{y \in \mathcal{S}_\sigma} \int_{[\sigma(y-),\sigma(y+)]} \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau \leq \int_0^1 \left| \frac{dU}{ds} \right| ds < \infty. \tag{4.20}$$

(iii) $\mathcal{C}_\sigma \subset \mathcal{C}_u$ *and* $\mathcal{S}_\sigma \supset \mathcal{S}_u$*.*

A generalized graph completely determines u and also specifies the paths connecting points of discontinuity and possible loops attached to the graph of u. There is no a priori mechanism, given u, for selecting a particular generalized graph. They may be induced by introducing paths at points of discontinuity in $\mathcal{S}_u$ (using straight lines [31] or families of Lipschitz paths [10]) and by possibly attaching loops at points of removable discontinuity or even at points of continuity in $\mathcal{C}_u$ (cf. the examples pointed out in [10] and the notion of extended graph in [27]). A converse to Proposition 4.7 has been proved by Raymond [27].

Proposition 4.8 (see [27]). *Given a function* $u : [a,b] \to \mathbb{R}^N$ *of bounded variation, a countable set* $\mathcal{T}$*, with* $[a,b] \supset \mathcal{T} \supset \mathcal{S}_u$*, and a family of Lipschitz paths* $\Phi = \{\phi_y\}_{y \in \mathcal{T}}$ *such that*

$$\begin{aligned} &\phi_y : [0,1] \to \mathbb{R}^N \text{ is Lipschitz continuous with} \\ &\phi_y(0) = u(y-),\ \phi_y(1) = u(y+) \quad \text{for } y \in \mathcal{T}, \end{aligned} \tag{A$_1$}$$

$$\sum_{y \in \mathcal{T}} \int_0^1 \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau < \infty, \tag{A$_2$}$$

there exists a generalized graph (X, U) *associated with the triplet* $\big(u, \mathcal{T}, \Phi\big)$*.*

The triplet $\big(u, \mathcal{T}, \Phi\big)$ is called extended graph in [27]. Apart from its theoretical interest, the proof of the proposition provides a procedure for constructing examples.

Proof. The construction proceeds in two steps.

Step 1. Construction of a continuous, BV representative $(Y, V) : [0,1] \to [a,b] \times \mathbb{R}^N$ of the graph determined by $\big(u, \mathcal{T}, \Phi\big)$.

Define $q : [a,b] \to [0,1]$ by setting $q(b) = 1$ and

$$\begin{aligned} q(x) &:= \frac{1}{Q} \left(x - a + \sum_{y \in \mathcal{T}, y < x} \int_0^1 \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau \right) \qquad \text{for } x \in [a,b), \\ Q &:= b - a + \sum_{y \in \mathcal{T}} \int_0^1 \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau. \end{aligned} \tag{4.21}$$

Then q is a strictly increasing left-continuous (but generally discontinuous) function satisfying the properties $\mathcal{C}_q = [a,b] \setminus \mathcal{T}$, $\mathcal{S}_q = \mathcal{T}$,

$$\begin{aligned} q(y+) - q(y-) &= \frac{1}{Q} \int_0^1 \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau, \\ 0 < x_2 - x_1 &\leq Q\big(q(x_2) - q(x_1)\big) \qquad \text{for } x_1 < x_2, \end{aligned} \tag{4.22}$$

and $q(a) = 0 < q(x) < 1 = q(b)$ for $x \in (a,b)$.

The domain and range of q admit the decompositions

$$[a,b] = \mathcal{C}_q \bigcup \mathcal{S}_q,$$
$$[0,1] = q(\mathcal{C}_q) \bigcup \Big(\bigcup_{y\in\mathcal{T}} [q(y-), q(y+)] \Big),$$

where $q(\mathcal{C}_q)$ and each $J_y = [q(y-), q(y+)]$ are mutually disjoint. The closure of the set $q(\mathcal{C}_q)$ is

$$\overline{q(\mathcal{C}_q)} = \Big(\bigcup_{y\in\mathcal{C}_q} \{q(y)\} \Big) \bigcup \Big(\bigcup_{y\in\mathcal{T}} \{q(y-), q(y+)\} \Big).$$

Define now the function (Y, V) as follows:

(a) On each interval $J_y = [q(y-), q(y+)]$ with $y \in \mathcal{T}$, set

$$\begin{cases} Y(s) = y & \text{for } s \in J_y, \\ V(s) = \phi_y\Big(\frac{s-q(y-)}{q(y+)-q(y-)}\Big) & \text{for } s \in J_y. \end{cases} \tag{4.23}$$

(b) On the complement $[0,1] - \cup_{y\in\mathcal{T}} J_y = q(\mathcal{C}_q)$, we have $s \in q(\mathcal{C}_q)$ if and only if $s = q(y)$ for precisely one $y \in \mathcal{C}_q$. We define

$$\begin{cases} Y(s) = y & \text{for } s \in q(\mathcal{C}_q), \\ V(s) = u(y) & \text{for } s \in q(\mathcal{C}_q). \end{cases} \tag{4.24}$$

Clearly, Y is an increasing function, (Y, V) are continuous on the interior of each interval J_y, and also for any $s_1, s_2 \in J_y$ with $s_1 < s_2$, we have

$$|V(s_2) - V(s_1)| \le \int_{\frac{s_1-q(y-)}{q(y+)-q(y-)}}^{\frac{s_2-q(y-)}{q(y+)-q(y-)}} \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau \le \int_0^1 \left| \frac{\partial \phi_y}{\partial \tau} \right| d\tau. \tag{4.25}$$

We proceed to show (Y, V) is continuous for each $s \in \overline{q(\mathcal{C}_q)}$. This follows by a case analysis:

(i) $s \in q(\mathcal{C}_q)$, $s_n \to s$ with $\{s_n\} \subset q(\mathcal{C}_q)$. Then $s_n = q(y_n)$, $s = q(y)$ for some $y_n, y \in \mathcal{C}_q$. By (4.22), $y_n \to y$ and thus

$$Y(s_n) = y_n \to y = Y(s), \quad V(s_n) = u(y_n) \to u(y) = V(s).$$

(ii) $s = q(y-)$ for some $y \in \mathcal{T}$, $s_n \to s$ with $\{s_n\} \subset q(\mathcal{C}_q)$. In this case for large n it is $s_n < s$, and the corresponding points $y_n \in \mathcal{C}_q$ satisfy $s_n = q(y_n)$ and $y_n < y$. Again (4.22) implies

$$0 < y - y_n < Q(q(y-) - q(y_n))$$

and thus, by (A_1),

$$Y(s_n) = y_n \to y- = Y(s), \quad V(s_n) = u(y_n) \to u(y-) = \phi_y(0) = V(s).$$

(iii) If $s = q(y+)$ for some $y \in \mathcal{T}$, $s_n \to s$ with $\{s_n\} \subset q(\mathcal{C}_q)$. Then, as in (ii),

$$Y(s_n) \to y+ = Y(s), \quad V(s_n) = u(y_n) \to u(y+) = \phi_y(1) = V(s).$$

(iv) Now let $s_n \to s$ with $\{s_n\} \subset \overline{q(\mathcal{C}_q)}$. For each n, let $\sigma_n \in q(\mathcal{C}_q)$ such that

$$|\sigma_n - s_n| < \frac{1}{n}, \quad |Y(\sigma_n) - Y(s_n)| < \frac{1}{n}, \quad |V(\sigma_n) - V(s_n)| < \frac{1}{n}.$$

Then $\sigma_n \to s$ and (i)–(iii) imply $Y(s_n) \to Y(s)$, $V(s_n) \to V(s)$.

(v) On the other extreme, let $s_n \to s$ with $\{s_n\} \subset \cup_{y \in \mathcal{T}} J_y$. Then $Y(s_n) = y_n$ with $y_n \in \mathcal{T}$. To simplify the exposition, consider the case $s_n < s$, $s_n \to s$. For each n, define $\sigma_n = q(y_n+)$. Then $\{\sigma_n\} \subset \overline{q(\mathcal{C}_q)}$, $\sigma_n \to s$ and (4.25) implies

$$\sum_n |V(\sigma_n) - V(s_n)| \le \sum_n \int_0^1 \left| \frac{\partial \phi_{y_n}}{\partial \tau} \right| d\tau.$$

It follows from (iv) and hypothesis (A_2) that $Y(s_n) = Y(\sigma_n) \to Y(s)$, $V(s_n) \to V(s)$.

(vi) For general sequences $s_n \to s$, the result follows by combining (iv) and (v).

The function Y is increasing and thus of bounded variation. The total variation of V may be explicitly computed

$$TV_{[0,1]}(V) \le TV_{[a,b]}(u) + \sum_{y \in \mathcal{T}} \left(\int_0^1 \left| \frac{\partial \phi_{y_n}}{\partial \tau} \right| d\tau - \big| u(y+) - u(y-) \big| \right).$$

Thus V is also of bounded variation.

Step 2. Construction of a Lipschitz continuous representative $(X, U) : [0,1] \to [a,b] \times \mathbb{R}^N$ of the graph determined by $\big(u, \mathcal{T}, \Phi\big)$.

Using the reparametrization (2.15) and the analysis of section 2.2, we can construct the canonical representative of the curve (Y, V). This representative (X, U) is Lipschitz (with Lipschitz constant L) and satisfies

$$(Y, V) = (X, U) \circ \gamma, \text{where } \gamma(s) = \frac{1}{L} T_{(Y,V)}(s),\ s \in [0,1], \text{ and } L = TV_{[0,1]}(Y, V)\ .$$

Also, X is increasing and (X, U) is a generalized graph. □

4.4. Comparison with definitions based on families of paths. In this section, we compare definitions based on families of paths with Theorem 3.4 based on generalized graphs.

We review the definition proposed by Dal Maso, LeFloch, and Murat [10]. This theory is based on a given family of Lipschitz continuous paths $\phi : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ that satisfy, for some $K > 0$ and for all u_0, $u_1 \in \mathbb{R}^N$ and $\tau \in [0,1]$, the properties

$$\phi(0; u_0, u_1) = u_0, \quad \phi(1; u_0, u_1) = u_1, \tag{H1}$$

$$\phi(\tau; u_0, u_0) = u_0, \tag{H2}$$

$$\left| \frac{\partial \phi}{\partial \tau}(\tau; u_0, u_1) \right| \le K\, |u_0 - u_1|. \tag{H3}$$

THEOREM AND DEFINITION 4.9 (see [10]). *Let $u : (a,b) \to \mathbb{R}^N$ be a function of bounded variation and $g : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous map. There exists a unique finite signed Borel measure μ on (a,b) such that*

(1) *if u is continuous on a Borel set $B \subset (a,b)$, then*

$$\mu(B) = \int_B g(u)\, du; \tag{4.26}$$

(2) *if u is discontinuous at a point $x \in (a,b)$, then*

$$\mu(\{x\}) = \int_0^1 g(\phi(\tau; u_-, u_+)) \frac{\partial \phi}{\partial \tau}(\tau; u_-, u_+) d\tau \qquad \text{with } u_\pm := u(x\pm). \tag{4.27}$$

The measure μ is called the nonconservative product of $g(u)$ by $\frac{du}{dx}$ and is denoted by

$$\mu = \Big[g(u) \frac{du}{dx} \Big]_\phi. \tag{4.28}$$

Remark 4.10. The stronger condition

$$\left| \frac{\partial \phi}{\partial \tau}(\tau; u_0, u_1) - \frac{\partial \phi}{\partial \tau}(\tau; v_0, v_1) \right| \le K \left| (u_0 - v_0) - (u_1 - v_1) \right| \tag{H3$'$}$$

is assumed in [10] in place of (H3), in connection with defining products of the form $g(u)\frac{dv}{dx}$, where u and v are BV functions. For instance, (H3$'$) guarantees that such products depend solely on the measure $\frac{dv}{dx}$ and not on the function v. It is straightforward to check that hypotheses (H1)–(H3) suffice for Definition 4.9, for most results presented in [10], and, in particular, for the theorem on weak stability.

It can be checked that the nonconservative product is independent of reparametrizations of the paths and that the definition is consistent with the usual distributional definition in the case of conservative products: if $f : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function, then

$$\Big[(Df)(u) \frac{du}{dx} \Big]_\phi = \frac{d}{dx}(f(u)). \tag{4.29}$$

The left-hand side in (4.29) is understood in the sense of Definition 4.9, while the right-hand side is understood in the sense of distributions.

Example 4.11. A simple example of paths is the *family of straight lines ϕ_S*, defined by

$$\phi_S(\tau; u_0, u_1) = u_0 + \tau(u_1 - u_0). \tag{4.30}$$

Then (4.27) reads

$$\Big[g(u) \frac{du}{dx} \Big]_S (\{x\}) = \left(\int_0^1 g(u_- + \tau(u_+ - u_-))\, d\tau \right) (u_+ - u_-),$$

and the nonconservative product coincides with a product introduced by Volpert [31]. To see that, recall that the *averaged superposition* of a BV function $u : (a,b) \to \mathbb{R}^N$ by a continuous function g is the function $\hat{g}(u)$, defined for all $x \in (a,b)$ by

$$\hat{g}(u)(x) = \int_0^1 g(u_- + s(u_+ - u_-))\, ds, \qquad u_\pm := u(x\pm). \tag{4.31}$$

Of course, we have

$$\hat{g}(u)(x) = g(u(x)) \quad \text{for all } x \in \mathcal{C}_u.$$

The function $\hat{g}(u)$ is Borel measurable, and the product $\hat{g}(u)\frac{du}{dx}$ is well defined as a signed Borel measure. This nonconservative product coincides with the one in Definition 4.9 if one uses the family of straight lines:

$$\left[g(u)\frac{du}{dx}\right]_S = \hat{g}(u)\frac{du}{dx} \tag{4.32}$$

as Borel measures on (a, b).

A comparison of (4.20) with the hypotheses of Theorem 4.9 indicates that (H2) and (H3) are somewhat restrictive, ruling out the possibility of loops attached to the graph of the BV function u. The gap between the two definitions has been bridged in a definition given by Raymond [27]. It is proved in [27] that this definition is equivalent to Definition 4.1.

In practice, there should be no confusion between the notation introduced in Definitions 4.1 and 4.9, respectively, in view of the following result.

THEOREM 4.12. *Let $u : (a, b) \to \mathbb{R}^N$ be a function of bounded variation and $(X, U) : [0, 1] \to [a, b] \times \mathbb{R}^N$ be a generalized graph of u. Suppose there exists a family of paths satisfying* (H1)–(H2) *such that, for every point of discontinuity $x \in \mathcal{S}_u$,*

$$\begin{aligned} &\phi(\tau; u_-, u+) := U\big(s_- + \tau(s_+ - s_-)\big), \qquad \tau \in [0, 1],\\ &\textit{where } \ s_\pm := X^{-1}(x\pm), \ \ u_\pm := u(x\pm), \end{aligned} \tag{4.33}$$

and satisfying the "no loop" condition

$$\textit{for every } \ x \in \mathcal{C}_u, \ \ \textit{there exists a unique } \ s \in [0, 1] \ \textit{ such that } \ X(s) = x. \tag{4.34}$$

Then the nonconservative products in Definitions 4.1 *and* 4.9, *respectively, coincide*

$$\left[g(u)\frac{du}{dx}\right]_\phi = \left[g(u)\frac{du}{dx}\right]_{(X,U)}$$

as Borel measures on (a, b).

Proof. It will be convenient to view the product in Definition 4.9 as a Borel–Stieltjes integral. Namely, by modifying $g(u)$ at most countably many points, we can construct a function $\overline{g(u)} : [a, b] \to \mathbb{R}^N$ satisfying

$$\overline{g(u)}(x) = g(u(x)) \quad \text{ for } \ x \in \mathcal{C}_u \tag{4.35}$$

and for $x \in \mathcal{S}_u$

(4.36)

$$\overline{g(u)} \cdot (u(x+) - u(x-)) = \int_0^1 g\big(\phi(\tau; u(x-), u(x+))\big)\partial_\tau \phi(\tau; u(x-), u(x+))\, d\tau.$$

Note that the value of $\overline{g(u)}(x)$ is not uniquely determined by points $x \in \mathcal{S}_u$, since any vector othogonal to the jump $u(x+) - u(x-)$ may be added to $\overline{g(u)}(x)$. From Definition 4.9, one deduces that

$$\left[g(u)\frac{du}{dx}\right]_\phi = \overline{g(u)}\, du \tag{4.37}$$

as Borel measures on (a, b), where the right-hand side is understood as a Borel–Stieltjes integral. Using the change of variable formula in Theorem 2.2, we thus have

$$\int_{[a,b]} \theta \left[g(u)\frac{du}{dx}\right]_\phi = \int_{[a,b]} \theta\, \overline{g(u)}\, du = \int_{[0,1]} (\theta \circ X)\, (\overline{g(u)} \circ X)\, d(u \circ X). \tag{4.38}$$

Consider the decomposition $\big([0,1]\setminus J_u\big)\cup J_u$ where $J_u := \bigcup_x [s_-,s_+]$ with $s_\pm := X^{-1}(x\pm)$. On one hand, by (4.35) one has $\overline{g(u)}(x) = g(u(x))$ for $x\in\mathcal{C}_u$ and thus, on the set $[0,1]\setminus J_u$, we obtain $u\circ X = U$ and $\overline{g(u)}\circ X = g(u\circ X) = g(U)$. Thus

$$\begin{aligned}\int_{[0,1]\setminus J_u} (\theta\circ X)\,(\overline{g(u)}\circ X)\,d(u\circ X) &= \int_{[0,1]\setminus J_u} (\theta\circ X)\,(g(u)\circ X)\,d(u\circ X)\\ &= \int_{[0,1]\setminus J_u} (\theta\circ X)\,g(U)\,\frac{dU}{ds}\,ds.\end{aligned}\tag{4.39}$$

On the other hand, in view of condition (4.34), each interval $[s_-,s_+]\subset J_u$ corresponds to a jump in u, say, $u_\pm := u(x\pm)$ for some $x\in\mathcal{S}_u$. Using (4.36) and (4.33), we obtain

$$\begin{aligned}\int_{[s_-,s_+]} (\theta\circ X)\,(\overline{g(u)}\circ X)\,d(u\circ X) &= \int_{[s_-,s_+]} \theta(x)\,\overline{g(u)}(x)\,d(u\circ X)\\ &= \theta(x)\,\overline{g(u)}(x)\cdot\big(u_+ - u_-\big)\\ &= \theta(x)\int_0^1 g\big(\phi(\tau;u(x-),u(x+))\big)\,\partial_\tau\phi\big(\tau;u(x-),u(x+)\big)\,d\tau\\ &= \theta(x)\int_{[s_-,s_+]} g(U(s))\,\frac{dU}{ds}\,ds.\end{aligned}\tag{4.40}$$

Combining (4.38)–(4.40) we deduce that

$$\begin{aligned}\int_{[a,b]} \theta\Big[g(u)\frac{du}{dx}\Big]_\phi &= \int_{[0,1]} (\theta\circ X)\,(\overline{g(u)}\circ X)\,d(u\circ X)\\ &= \int_{[0,1]} (\theta\circ X)\,g(U)\,\frac{dU}{ds}\,ds\\ &= \int_{[a,b]} \theta\Big[g(u)\frac{du}{dx}\Big]_{(X,U)}\end{aligned}$$

for every test function θ. □

Next, we list examples in order to illustrate the relation between regularized sequences $\{v_n\}$, subject to (1.3), and the associated nonconservative products. Since all definitions of nonconservative products are equivalent within their range of applicability, we will use interchangeably the notation $[g(u)\frac{du}{dx}]_\phi$ and $[g(u)\frac{du}{dx}]_{(X,U)}$; the former is applicable when we are given a family of paths ϕ or an extended graph and the latter when we are given a generalized graph (X,U). In any case one can pass from ϕ to (X,U) and vice versa by using Propositions 4.7 and 4.8. We recall that, given a generalized graph (X,U), it is always possible to construct a sequence of smooth functions $\{v_n\}$ that approach (X,U) in the graph distance (cf. Theorem 3.2).

Example 4.13. We return to the sequence $\{v_n\}$ discussed in Example 1.1. With u_0, $u_1\in\mathbb{R}^N$, the functions $\{v_n\}$, v, and the path π defined in (1.5)–(1.6), we have

$$g(v_n)\frac{dv_n}{dx} \rightharpoonup \left(\int_0^1 g(\pi(s))\pi'(s)ds\right)\delta_{x_0} \quad \text{weak-}\star \text{ in } \mathcal{M}[a,b].\tag{4.41}$$

We select the family of paths ϕ so that $\phi(.;u_0,u_1)=\pi$ holds. Then the nonconservative product reads

$$\Big[g(v)\frac{dv}{dx}\Big]_\phi = c(g,\pi)\,\delta_{x_0}, \qquad \text{where } c(g,\pi) = \int_0^1 g(\pi(s))\pi'(s)ds\ ,\tag{4.42}$$

and for any g continuous we have $g(v_n)\frac{dv_n}{dx} \rightharpoonup [g(v)\frac{dv}{dx}]_\phi$ weak-$\star$. This example illustrates the important fact that the path ϕ must be selected in agreement with the regularization under consideration. Different regularizations may give rise to different paths ϕ.

When $u_0 \neq u_1$, (4.42) may be interpreted in terms of Definition 4.9. When $u_0 = u_1$, this is no longer possible, because hypothesis (H_2) excludes the possibility of loops. However, it can be interpreted in terms of the more general Definition 4.1 as follows: By Proposition 4.8 the function v together with the location of the loop discontinuity x_0 and the path π determine a generalized graph. A representative (Y,V) of this graph is given by the formulas

(4.43)
$$Y(s) = \begin{cases} a + 3s(x_0 - a), & s \in [0, \frac{1}{3}], \\ x_0, & s \in [\frac{1}{3}, \frac{2}{3}], \\ x_0 + (3s-2)(b - x_0), & s \in [\frac{2}{3}, 1], \end{cases} \qquad V(s) = \begin{cases} u_0, & s \in [0, \frac{1}{3}], \\ \pi(3\,s - 1), & s \in [\frac{1}{3}, \frac{2}{3}], \\ u_1, & s \in [\frac{2}{3}, 1]. \end{cases}$$

Then Definition 4.1 gives

$$\Big[g(v)\frac{dv}{dx}\Big]_{(Y,V)} = c(g;\pi)\,\delta_{x_0} \tag{4.44}$$

and so, as $n \to \infty$,

$$g(v_n)\frac{dv_n}{dx} \rightharpoonup \Big[g(v)\frac{dv}{dx}\Big]_{(Y,V)} \qquad \text{weakly-}\star \in \mathcal{M}[a,b]. \tag{4.45}$$

Note that (4.44) holds for arbitrary u_0 and u_1 and that, when $u_0 = u_1$, the limiting graph (Y,V) contains a loop at the location x_0. $\square$

Example 4.14. Consider next a piecewise constant function $v : [a,b] \to \mathbb{R}^N$ having three points of discontinuity:

$$v(x) = \begin{cases} u_0 & \text{for } x \in [a, c_1), \\ u_1 & \text{for } x \in [c_1, c_2), \\ u_2 & \text{for } x \in [c_2, c_3), \\ u_3 & \text{for } x \in [c_3, b], \end{cases} \tag{4.46}$$

where $a < c_1 < c_2 < c_3 < b$ are real constants, and the u_j's are constant vectors. Let π_j be Lipschitz continuous paths such that $\pi_j(0) = u_{j-1}$ and $\pi_j(1) = u_j$, $j = 1,2,3$. In a fashion similar to Example 1.1, we can define a sequence of smooth functions v_n by replacing the jumps in v with smooth transition layers based on the paths π_j such that $\{v_n\}$ are uniformly bounded and $v_n \to v$ pointwise. Then

$$g(v_n)\frac{dv_n}{dx} \rightharpoonup \sum_{j=1,2,3} c(g,\pi_j)\,\delta_{c_j}\,, \qquad \text{where} \quad c(g,\pi_j) = \int_0^1 g\big(\pi_j\big)\frac{\partial \pi_j}{\partial s}\,ds\,. \tag{4.47}$$

Accordingly, the nonconservative product is defined so that

$$\Big[g(v)\frac{dv}{dx}\Big]_{(X,U)} = \Big[g(v)\frac{dv}{dx}\Big]_\phi = \sum_{j=1,2,3} c(g,\pi_j)\,\delta_{c_j}. \tag{4.48}$$

In most cases this is done by using Definition 4.9, upon selecting the family of paths ϕ so that $\phi(.;u_j,u_{j+1}) = \pi_j$. There are a few interesting exceptions when one needs to use Definition 4.1. One is the case where the approximating sequence contains loops. This is discussed in the previous example. Another case is when the jumps of v at $x = c_1$ and $x = c_3$ coincide, $u_0 = u_2$ and $u_1 = u_3$. Then Definition 4.9 prevents us from using, in v_n, different paths for approximating the same jump at two different locations. This difficulty does not arise with Definition 4.1. Upon constructing a representative of the limiting graph, as in the previous example, we define the nonconservative product as in (4.48). □

Example 4.15. Given an increasing sequence of points $c_k \in [a, b)$, $k = 0, 1, 2, \dots,$ with $c_0 = a$ and $c_\infty := \lim_{k\to\infty} c_k \in (a, b)$, we consider the saltus function $u : [a, b] \to \mathbb{R}^N$ defined by

$$v(x) = \begin{cases} u_0 & \text{for } x \in [a, c_1), \\ u_k & \text{for } x \in [c_k, c_{k+1}), \quad k = 1, 2, \dots, \\ u_\infty & \text{for } x \in [c_\infty, b], \end{cases} \tag{4.49}$$

for constants u_k and u_∞ in $\mathbb{R}^N$. For each jump connecting u_k to u_{k+1} in v, we consider a Lipschitz continuous path $\pi_k(s)$ for $s \in [0, 1]$ satisfying $\pi_k(0) = u_{k-1}$ and $\pi_k(1) = u_k$.

Let $c_k^{\pm,n}$, for $k, n = 1, 2, \dots,$ be a sequence of points in the interval (a, b) such that $c_k^{-,n} < c_k < c_k^{+,n} < c_{k+1}^{-,n}$, and $c_k^{\pm,n} \to c_k$ as $n \to \infty$. We construct the sequence of regularized functions $v_n : [a, b] \to \mathbb{R}^N$ by

$$v_n(x) = \begin{cases} u_0 & \text{for } x \in [a, c_1^{-,n}), \\ \pi_k\left(\frac{x - c_k^{-,n}}{c_k^{+,n} - c_k^{-,n}}\right) & \text{for } x \in [c_k^{-,n}, c_k^{+,n}), \quad k = 1, 2, \dots, \\ u_k & \text{for } x \in [c_k^{+,n}, c_{k+1}^{-,n}), \quad k = 1, 2, \dots, \\ u_\infty & \text{for } x \in [c_\infty, b]. \end{cases} \tag{4.50}$$

The functions v_n are continuous and

$$TV(v_n) = \sum_{k=1}^{\infty} \int_0^1 \left| \frac{d\pi_k}{ds}(s) \right| ds. \tag{4.51}$$

We assume that the right-hand side of (4.51) is finite, so that the sequence $\{v_n\}$ is of uniformly bounded variation. A calculation shows that

$$g(v_n)\frac{dv_n}{dx} \rightharpoonup \sum_{j=1}^{\infty} c(g, \pi_j)\, \delta_{c_j}\,, \qquad \text{weak-}\star \text{ in } \mathcal{M}[a, b], \tag{4.52}$$

which suggests that the nonconservative product $[g(v)\frac{dv}{dx}]_{(X,U)}$ should be defined by

$$\Big[g(v)\frac{dv}{dx}\Big]_{(X,U)} = \sum_{k=1}^{\infty} c(g, \pi_k)\, \delta_{c_k}\,. \tag{4.53}$$

Because of the uniform constant K in hypothesis (H_3) this cannot be handled in general by Definition 4.9. By contrast, Definition 4.1 is adequate to define the nonconservative product as in (4.53); this follows from (A_1)–(A_2) and the construction process in the proof of Proposition 4.8. □

5. The Riemann problem for nonconservative hyperbolic systems. The theory developed in the previous sections is now applied to the Riemann problem for first-order quasi-linear hyperbolic systems

$$\partial_t u + A(u)\,\partial_x u = 0, \qquad x \in \mathbb{R},\ t > 0\,, \tag{5.1}$$
$$u(x,0) = \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x > 0, \end{cases}$$

where the $N \times N$ matrix $A(u)$ is a smooth function of u, and u_+ and u_- are given vectors in $\mathbb{R}^N$. Because of the invariance of the Riemann problem under dilations $(x,t) \to (\alpha x, \alpha t)$, for $\alpha > 0$, the solution is expected to be a self-similar function of the variable $\xi = x/t$. Accordingly, $u = u(x/t)$ is sought by solving the boundary value problem

$$\begin{aligned} -\xi\,\frac{du}{d\xi} + A(u)\,\frac{du}{d\xi} &= 0, \\ u(\pm\infty) &= u_\pm\,. \end{aligned} \tag{P$_0$}$$

In the nonconservative case $A(u)$ is not a Jacobian matrix, and one is confronted with the difficulty of giving an appropriate meaning to the product $A(u)\,du/d\xi$. To address this difficulty, we construct solutions of (P$_0$) as $\varepsilon \searrow 0$ limits of solutions to

$$\begin{aligned} -\xi\,\frac{du_\varepsilon}{d\xi} + A(u_\varepsilon)\,\frac{du_\varepsilon}{d\xi} &= \varepsilon\,\frac{d}{d\xi}\left(B(u_\varepsilon)\,\frac{du_\varepsilon}{d\xi}\right), \\ u_\varepsilon(\pm\infty) &= u_\pm\,, \end{aligned} \tag{P$_\varepsilon$}$$

where $B(u)$ is a positive semidefinite $N \times N$ matrix. This approach for constructing solutions to the Riemann problem is called self-similar zero-viscosity limits. For conservative strictly hyperbolic systems, this method is known to select shocks having the internal structure of a traveling wave and to provide the unique solution to the Riemann problem for weak waves; see Tzavaras [30].

Throughout the paper we proceed under the hypothesis: There exists a family of smooth solutions u_ε to (P$_\varepsilon$), for $\varepsilon > 0$, that satisfy *uniform in* ε L^∞ and variation bounds,

$$|u_\varepsilon - u_-|_{L^\infty} + TV(u_\varepsilon) \,\le\, C\,, \tag{5.2}$$

as well as uniform convergence properties at infinity,

$$|u_\varepsilon(\xi) - u_\pm| \le C\exp(-\alpha\,\varepsilon) \qquad \text{for } \xi < a+1 \text{ and } \xi > b-1, \tag{5.3}$$

for some $a < b$ and $C, \alpha > 0$ independent of ε. Such solutions are constructed in LeFloch–Tzavaras [18, 19] under the following set of structural assumptions:

(i) System (1.1) is strictly hyperbolic; i.e., the matrix $A(u)$ has N real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$;

(ii) the initial jump $|u_+ - u_-|$ is sufficiently small;

(iii) the diffusion matrix $B(u)$ is the $N \times N$ identity matrix Id.

It is, however, expected that estimates (5.2) should hold, together with (5.3) or variants, under more general circumstances, and the analysis in this section requires only (5.2)–(5.3).

The uniform BV estimates provide a natural framework to study the notion of weak solutions for the nonconservative Riemann problem (P_0). Let $(X_\varepsilon, U_\varepsilon)$ be the arc-length reparametrization of the graph $(\xi, u_\varepsilon(\xi))$. Theorem 3.2 asserts that there exists a subsequence $\{u_{\varepsilon_n}\}$ and a generalized graph (X, U), determining a function u of bounded variation such that

$$
\begin{aligned}
&(X_{\varepsilon_n}, U_{\varepsilon_n}) \xrightarrow{\mathrm{d}} (X, U),\\
&\sigma_{\varepsilon_n}(\xi) \to \sigma(\xi)\,, \quad u_{\varepsilon_n}(\xi) \to u(\xi) \quad \text{for } \xi \in \mathcal{C}_\sigma .
\end{aligned}
\tag{5.4}
$$

Recall that $\sigma_{\varepsilon_n} = X_{\varepsilon_n}^{-1}$ is a strictly increasing function, while $\sigma = X^{-1}$ is a strictly increasing multivalued map.

Using the results of sections 3 and 4, we can give a meaning to the nonconservative product $[A(u)\frac{du}{d\xi}]_{(X,U)}$, relative to the generalized graph (X, U), as a weak-$\star$ limit of $A(u_\varepsilon)\frac{du_\varepsilon}{dx}$. To this end, we use either Definition 4.1 to interpret $[A(u)\frac{du}{d\xi}]_{(X,U)}$ as a Radon measure or Definition 4.4 to define it via its distribution function F. It leads to a notion of solutions for $(\mathcal{P}_0)$ as in Definition 5.1.

Definition 5.1. *Let $(X, U)\ : [0, 1] \to [a, b]\times\mathbb{R}^N$ be a generalized graph associated with a function of bounded variation $u : [a, b] \to \mathbb{R}^N$. We say that (X, U) is a* weak solution *to the system*

$$
-\xi \,\frac{du}{d\xi} + A(u)\,\frac{du}{d\xi} = 0
\tag{5.5}
$$

if

$$
-\xi \,\frac{du}{d\xi} + \Big[A(u)\frac{du}{d\xi}\Big]_{(X,U)} = 0
\tag{5.6}
$$

in the sense of measures. Equivalently, if for any $\zeta,\, \xi \in [a, b]$,

$$
-\Big[\xi\, u(\xi+) - \zeta\, u(\zeta-)\Big] + \int_\zeta^\xi u(\theta)d\theta + \int_{X^{-1}(\zeta-)}^{X^{-1}(\xi+)} A(U(s))\frac{dU}{ds}\, ds = 0.
\tag{5.7}
$$

Remark 5.2. Relation (5.7) suggests that at points $\xi \in \mathcal{S}_{X^{-1}}$, the set where the inverse map X^{-1} is multivalued, the following analogue of the Rankine–Hugoniot conditions is satisfied:

$$
-\xi\Big[u(\xi+) - u(\xi-)\Big] + \int_{X^{-1}(\xi-)}^{X^{-1}(\xi+)} A(U(s))\frac{dU}{ds}\, ds = 0\,.
\tag{5.8}
$$

Points $\xi \in \mathcal{S}_{X^{-1}}$ may correspond either to jumps or to loops.

The notion of weak solution depends on the equivalence class, but not on the specific representative, of the generalized graph.

Proposition 5.3. *If a generalized graph (X, U) is a weak solution to* (5.4)*, then any path (Y, V) belonging to the same equivalence class as (X, U) is also a weak solution.*

Suppose there exists $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $A = Df$. Let u of bounded variation be a solution to (5.4) *in the sense of distributions*

$$
\int_{\mathbb{R}} u\,\theta\, d\xi + \int_{\mathbb{R}} \big(\xi\, u + f(u)\,\big)\,\frac{d\theta}{d\xi}\, d\xi = 0
\tag{5.9}
$$

for every smooth function θ of compact support. Then any generalized graph associated with u is a weak solution in the sense of Definition 5.1.

The proof of Proposition 5.3 follows from the facts that the nonconservative product is independent of reparametrizations of the generalized graph (X, U), and, when $A(u) = Df(u)$, one has

$$\Big[A(u)\frac{du}{d\xi}\Big]_{(X,U)} = \frac{d}{d\xi} f(u)$$

as measures.

THEOREM 5.4. *Fix $u_\pm \in \mathbb{R}^N$. Let $u_\varepsilon : (-\infty, +\infty) \to \mathbb{R}^N$ be a family of smooth solutions to (P_ε) for $\varepsilon > 0$ that are of uniformly bounded variation and satisfy* (5.2)–(5.3)*. Consider the arc-length parametrizations $(X_\varepsilon, U_\varepsilon)$ of the graphs of u_ε. There exists a subsequence $\{u_{\varepsilon_n}\}$, with $\varepsilon_n \to 0$, a generalized graph (X, U), and an associated BV function u, such that $(X_{\varepsilon_n}, U_{\varepsilon_n})$ converges to (X, U) as in* (5.4)*, (X, U) is a weak solution of* (5.5)*, and*

$$u(\xi) = \begin{cases} u_- & \textit{for } -\infty < \xi < a + 1, \\ u_+ & \textit{for } b - 1 < \xi < +\infty. \end{cases} \tag{5.10}$$

Combined with [18, 19], where the uniform bounds are established for strictly hyperbolic systems and small initial jumps $|u_+ - u_-|$, Theorem 5.4 provides an existence result for the Riemann problem (P_0). We refer to [19] for the structure of the resulting wave-fan solution of the Riemann problem and the admissibility restrictions that the process (P_ε) imposes on shocks. The relation with the solution of the Riemann problem for genuinely nonlinear systems, obtained in [14, 10], is also investigated in [19].

Proof. In view of the uniform estimates (5.2)–(5.3) and Theorem 3.2, the graphs $(X_\varepsilon, U_\varepsilon)$ converge along subsequences in the graph distance. Denote by (X, U) the limiting graph.

Observe that the right-hand side of the equation in (P_ε) tends to zero in the sense of distributions

$$\left| \varepsilon \int B(u_\varepsilon)\, \frac{du_\varepsilon}{d\xi}\, \frac{d\theta}{d\xi}\, dx \right| \le \varepsilon\, C \|\theta\|_{\mathcal{C}^1}\, TV(u_\varepsilon) \quad \longrightarrow 0 \tag{5.11}$$

for every test function θ.

To determine the limit of the right-hand side of the equation in (P_ε), one writes

$$A(u_\varepsilon)\, \frac{du_\varepsilon}{d\xi} = \Big[A(u_\varepsilon)\frac{du_\varepsilon}{d\xi}\Big]_{(X_\varepsilon, U_\varepsilon)}$$

and one uses the weak stability theorems, either Theorem 4.3 if the nonconservative product is viewed as a Radon measure or Theorem 4.6 if the distribution function is used instead. It follows that

$$\Big[A(u_\varepsilon)\frac{du_\varepsilon}{d\xi}\Big]_{(X_\varepsilon, U_\varepsilon)} \rightharpoonup \Big[A(u)\frac{du}{d\xi}\Big]_{(X,U)} \quad \text{weak-}\star \text{ in } \mathcal{M}[a, b].$$

Using (5.11), we conclude that (X, U) is a weak solution in the sense of Definition 5.1. Finally, the fact that $u(\xi)$ admits the boundary conditions as in (5.10) is a direct consequence of (5.3). ∎

Acknowledgment. The support of this joint project by the TMR project HCL ERBFMRXCT 960033 is gratefully acknowledged.

REFERENCES

[1] P. Aviles and Y. Giga, *Variational integrals on mappings of bounded variation and their semicontinuity*, Arch. Rational Mech. Anal., 115 (1991), pp. 201–255.

[2] L Boccardo and F. Murat, *Remarques sur l'homogénéisation de certains problèmes quasi-linéaires*, Portugal. Math., 41 (1982), pp. 535–562.

[3] A. Bressan and F. Rampazzo, *On differential systems with vector-valued impulsive controls*, Boll. Un. Mat. Ital., 7 (1988), pp. 641–656.

[4] F. Bouchut and F. James, *One-dimensional transport equations with discontinuous coefficients*, Nonlinear Anal., 32 (1997), pp. 891–933.

[5] F. Cardin and G. Zanzotto, *On continuum theories involving quasilinear first order nonconservative systems with involutions and a supplementary inequality*, Contin. Mech. Thermodyn., 3 (1991), pp. 53–63.

[6] J.F. Colombeau, *Elementary Introduction to New Generalized Functions*, North–Holland, Amsterdam, 1985.

[7] J.F. Colombeau and A.Y. Leroux, *Multiplications of distributions in elasticity and hydrodynamics*, J. Math. Phys., 29 (1988), pp. 315–319.

[8] C. Cordier, P. Degond, P. Markowich, and C. Schmeiser, *Travelling wave analysis and jump relations for Euler-Poisson model in the quasineutral limit*, Asymptot. Anal., 11 (1995), pp. 209–240.

[9] C.M. Dafermos, *Solution of the Riemann problem for a class of hyperbolic conservation laws by the viscosity method*, Arch. Rational Mech. Anal., 52 (1973), pp. 1–9.

[10] G. Dal Maso, P.G. LeFloch, and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., 74 (1995), pp. 483–548.

[11] G.B. Folland, *Real Analysis. Modern Techniques and Their Applications*, Wiley-Interscience, New York, 1984.

[12] T.Y. Hou and P.G. LeFloch, *Why nonconservative schemes converge to wrong solutions: error analysis*, Math. Comp., 62 (1994), pp. 497–530.

[13] P.D. Lax, *Hyperbolic systems of conservation laws* II, Comm. Pure Appl. Math., 10 (1957), pp. 537–566.

[14] P.G. LeFloch, *Entropy weak solutions to nonlinear hyperbolic systems in nonconservative form*, Comm. Partial Differential Equations, 13 (1988), pp. 669–727.

[15] P.G. LeFloch, *Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form*, Report 593, Institute for Mathematics and Its Applications, Minneapolis, MN, 1989.

[16] P.G. LeFloch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, in Nonlinear Evolution Equations that Change Type, B. Keyfitz and M. Shearer, eds., IMA Vol. Math. Appl. 27, Springer, New York, 1990, pp. 126–138.

[17] P.G. LeFloch and T.P. Liu, *Existence theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., 5 (1993), pp. 261–280.

[18] P.G. LeFloch and A.E. Tzavaras, *Existence theory for the Riemann problem for nonconservative hyperbolic systems*, C. R. Acad. Sci., Paris Sér. I Math., 323 (1996), pp. 347–352.

[19] P.G. LeFloch and A.E. Tzavaras, in preparation.

[20] P.G. LeFloch and Z.P. Xin, *Uniqueness via the adjoint problems for systems of conservation laws*, Comm. Pure Appl. Math., 46 (1993), pp. 1499–1533.

[21] M. Marcus and V.J. Mizel, *Absolute continuity on tracks and mapping of Sobolev spaces*, Arch. Rat. Mech. Anal., 45 (1972), pp. 297–320.

[22] F. Poupaud and M. Rascle, *Measure solutions to the linear multidimensional transport equation with discontinuous coefficients*, Comm. Partial Differential Equations, 22 (1997), pp. 337–358.

[23] V.H. Ransom and D.L. Hicks, *Hyperbolic two-pressure models for two-phase flow*, J. Comput. Phys., 53 (1984), pp. 124–151.

[24] P.-A. Raviart, *Short Course on Nonlinear Hyperbolic Systems and Numerical Analysis*, Anogia Workshop, Crete, Greece, 1995, unpublished notes.

[25] P.-A. Raviart and L. Sainsaulieu, *A nonconservative hyperbolic system modeling spray dynamics,* I. *Solution of the Riemann problem*, Math. Models Methods Appl. Sci., 5 (1995), pp. 297–333.

[26] J.P. Raymond and D. Seghir, *Lower semicontinuity and integral representation of functionals*

in $BV([a,b];R^m)$, J. Math. Anal. Appl., 188 (1994), pp. 956–984.
[27] J.P. Raymond, *Definition and Stability of Nonconservative Products*, 1995, preprint.
[28] L. Sainsaulieu, *An Euler system modeling vaporizing sprays*, in Dynamics of Heterogeneous Combustion and Reacting Systems, Progress Astr. Aero., 152 (1993), pp. 280–305.
[29] L. Tartar, *Compensated compactness and applications to partial differential equations*, in Nonlinear Analysis and Mechanics, Heriot Watt Symposium, Vol. IV, R.J. Knops, ed., Pitman Research Notes in Math., New York, 1979, pp. 136–192.
[30] A.E. Tzavaras, *Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws*, Arch. Rational Mech. Anal., 135 (1996), pp. 1–60.
[31] A.I. Volpert, *The space* BV *and quasilinear equations*, Math. Sbornik, 73 (1967), pp. 225–267.