Bifurcation and creep effects in a viscoelastic non-local damageable continuum

Theocharis Baxevanis\textsuperscript{a}, Gilles Pijaudier-Cabot\textsuperscript{b}, Frédéric Dufour\textsuperscript{b,∗}

\textsuperscript{a} Department of Applied Mathematics, University of Crete, Heraklion, 714 09 Heraklion, Greece

\textsuperscript{b} ERT R&DO, Institut de Recherche en Génie Civil et Mécanique, Centrale Nantes, CNRS, Université de Nantes, 1, rue de la noë, BP 92101, 44321 Nantes cedex 3, France

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Abstract

The conditions for localization in a material described by a non-local damage-based constitutive relation coupled with a Kelvin type creep relation are derived in a closed form. The inception of a localized mode is considered as a bifurcation into a harmonic mode. The criterion of bifurcation is reduced to the classical form of singularity of a pseudo acoustic tensor; this tensor involves the elasto-damage strain and the total one at the inception of localization and the wavelength of the bifurcation mode through the Fourier transform of the weight function used in the definition of non-local damage. A geometrical approach was adopted to analyze localization for loading paths such that the elastic strain tensor is a fraction of the total strain tensor. Such loading paths include the general triaxial ones for which changes in the loading state occur only under time-independent processes (negligible creep strain during these changes of the stress state) and the uniaxial loading. The proposed coupled model preserves the properties of localization limiters; the minimum wavelength of the localization modes cannot be zero. The critical wavelength which is related to the width of the localization zone increases when the material parameter \( \alpha (0 \leq \alpha \leq 1) \), which is the fraction of creep strain entering into the evolution of damage, is decreasing. Under a certain condition on the growth of the loading function of damage and the initial state of deformation the critical wavelength decreases as the creep effect (creep strains) increases in accordance with experimental observations—increase of brittleness due to creep. In uniaxial tension, and for a specific yield function of concrete considered in this paper, this condition is fulfilled whenever the initial damage is in a region near the first occurrence of localization.

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1. Introduction

The growth of micro-cracks in progressively fracturing rate-independent materials is a fairly distributed process which is associated to stable material response. For certain stress trajectories, however, a different deformation mode

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\textsuperscript{∗∗} Corresponding author.

E-mail addresses: theocharis@tem.uoc.gr (T. Baxevanis), Gilles.Pijaudier-Cabot@ec-nantes.fr (G. Pijaudier-Cabot), Frederic.Dufour@ec-nantes.fr (F. Dufour).

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may prevail, consisting of the formation of discrete failure planes (macro-cracks). In the latter case, the mechanical response as observed at the macro-scale becomes, in general, unstable. The inception of a localized mode may be considered as a bifurcation problem related to the loss of positive definiteness of the tangent material stiffness operator governing the homogeneous deformation (Rudnicki and Rice, 1975; Rice, 1976). In mathematical terms, the elliptic character of the set of partial differential equations governing equilibrium is lost (ill-posedness of the related boundary value problem). This result was derived for the linearized rate equation problem, considering what is commonly denoted as a linear comparison solid (Hill, 1959). Loss of ellipticity may result in a discontinuous rate of deformation and constant total energy consumption. Path stability considerations indicate that this particular solution is expected and consequently failure occurs without energy dissipation (Bazant, 1988). In dynamics, the situation is similar. The hyperbolic differential equations of motion may become elliptic and consequently the initial-boundary value problem ill-posed (Hadamard, 1903).

Non-conventional constitutive relations, called localization limiters, based on sophisticated techniques that enrich the standard continuum, have been used by many authors to eliminate the deficiencies caused by ill-posedness of boundary value problems. A wide class of localization limiters is based on the assumption that the stress at a material point is not only determined by the history of strain at this point but also by the interactions with other material points (non-local integral continuum). The distance over which interactions are important is related to an internal length that prevents energy dissipation to become zero as failure occurs (Simo and Fox, 1989; Bazant and Pijaudier-Cabot, 1988). Another class of localization limiters relies on enrichment by terms that contain gradients of the state variables. Some of these models deal with strain gradients, i.e. higher-order gradients of the displacement field. In the above theories the rate-independent nature of the material model is preserved. Incorporation of rate-dependent viscous terms can limit localization due to softening (Needleman, 1988; Loret and Prevost, 1990; Dubé et al., 1996). The size of the localization zone, however, is controlled by this rate effect and it may not be possible to fit, with the same expression of the evolution of damage or viscoplastic strains, the rate effect on the stress–strain response and a width of the localization zone that is consistent with experimental observations.

In this contribution, attention is focused on concrete structures and we are interested in predicting the creep failure due to localization of damage of those structures. In common practice, it is usually assumed for concrete that linear viscoelasticity takes place for low stress levels where the instantaneous mechanical behavior is elastic. In the contrary, for high stress levels the linearity hypothesis fails. Bazant (1993) has shown that creep at high stress levels cannot be captured merely by a non-linear generalization of viscoelastic stress–strain relation. Experimental evidence associate creep strains at high stress levels to microcracks nucleation and growth, i.e. to damage (Rüsch was the first one to conduct such experiments on concrete structures (Rüsch, 1960)). As recalled by Bazant and Planas (1998), time-dependent fracture of concrete is caused by viscoelasticity of the material, and (or) breakage of bonds in the fracture process zone. The difference between these two effects can be observed on structural responses and size effect. In the range of quasi-static loading, that is, in the absence of inertia forces and wave propagation effects, viscoelasticity in the bulk (linear creep) causes an increase of the bearing capacity for increasing loading rates, with a decreasing peak displacement. For geometrically similar structures of different sizes and mode I crack propagation, data points on the size effect plot shift toward increasing brittleness for decreasing loading rates. On this size effect plot, one finds on the vertical axis the nominal stress (at peak load) divided by the tensile strength and on the horizontal axis the brittleness coefficient – the ratio of the structure size to a material representative size, and on the log–log plot of $−1/2$. It corresponds to a decrease of the size of the fracture process zone compared to the size of the structure. Such a shift has been observed experimentally by Bazant and Gettu (1992), Bazant and Li (1997), as shown in Fig. 1. On the other hand, if the rate dependence is caused only by bond breakages, the peak loads correspond to increasing displacements and there is no shift of this kind on the size effect plot. Note that ductility and brittleness are defined here, and in the remaining, as the position of the experimental data with respect to the two asymptotic limits on the size effect plot: strength of materials and LEFM (see Fig. 1). Ductility increases when the set of data points shift to the left and brittleness increases when they shift to the right.

Many authors have used coupled models to capture mechanical, viscoelastic and non-linear instantaneous behaviors. In these models, linear viscoelasticity is coupled to a rate independent elasto-plastic model (de Borst et al., 1993), to a smeared crack model (Rots, 1988) or to a damage model (Mazzotti and Savoia, 2003). The aim of this paper is to investigate the condition of localization for such a coupled model and thus to see the influence of creep on failure.
In the present study, we will consider as an example a viscoelastic model coupled to rate independent damage inspired from Omar (2004), in which the relationship between the effective stress, defined in a standard way according to continuum damage (Lemaitre and Chaboche, 1985), and the strain follows a Kelvin chain. The analysis follows the method used by Benallal (1992) for thermo-mechanical problems and Pijaudier-Cabot and Benallal (1993) for the case of a rate independent non-local model. This paper is organized as follows: In Section 2, the creep and damage models are briefly recalled and the rate formulation for creep-damage model is derived based on the small strain assumption. In Section 3, the strain localization analysis is performed. The rate constitutive relations for the linear comparison solid, as well as the bifurcation conditions under the assumption that the solid considered is either infinite or at least large enough so that boundary layers effects can be neglected, are derived. Moreover, the dependence of the admissible wavelengths on the parameters of constitutive law and on the state variables is discussed. An illustration is presented in the case of a uniaxial response in Section 4.

2. Constitutive relation

2.1. Creep model

Consider first a quasi-brittle material, such as concrete, exhibiting non-ageing linear viscoelasticity and characterized by the uniaxial compliance function $J(t, t')$, representing the uniaxial strain at age $t$ caused by a unit stress enforced at any age $t'$ (the linearity hypothesis agrees very well with test results in which no strain reversals have taken place, in the case of basic creep, and for stress levels less than 40% of the strength limit). The total strain reads:

$$
\varepsilon_{\text{tot}}(t) = J(t, t') \sigma(t) + \int_{t'}^{t} J(t, \tau) d\sigma(\tau).
$$

The assumption of a non-ageing material will be used throughout this paper for the sake of simplicity, and causes $J(t, t')$ to be a function of just the time lag $(t - t')$.

By approximating the compliance function by a Dirichlet series

$$
J(t - t') \approx \frac{1}{E} + \sum_{i=1}^{N} \frac{1}{E_i} \left[ 1 - \exp\left(\frac{-t - t'}{\tau_i}\right) \right],
$$

(2)
where $\tau_i, i = 1, 2, \ldots, N$, are fixed parameters called retardation times, $E$ is Young’s modulus and $E_i, i = 1, 2, \ldots, N$, are age-independent moduli which can be determined by least-square fitting to the “exact” compliance function, it can be proved that

$$\epsilon^{\text{tot}}(t) = \epsilon^{el,d}(t) + \sum_{i=1}^{N} \epsilon_i^{cr}(t), \quad i = 1, 2, \ldots, N,$$

(3)

where the strains $\epsilon^{el,d}$ and $\epsilon_i^{cr}, i = 1, 2, \ldots, N$, are governed by the following equations

$$E \epsilon^{el,d}(t) = \sigma(t),$$

(4)

and

$$E_i \epsilon_i^{cr}(t) + \tau_i E_i \dot{\epsilon}_i^{cr}(t) = \sigma(t), \quad i = 1, 2, \ldots, N.$$  

(5)

The total stress $\sigma$ in (5) is expressed as the sum of two terms. The first term, $E_i \epsilon_i^{cr}$, corresponds to the stress of an elastic spring of stiffness $E_i$. The second term, $\tau_i E_i \dot{\epsilon}_i^{cr}$, is the stress generated by the strain rate $\dot{\epsilon}_i^{cr}$ in a linear dashpot of viscosity $\eta_i = \tau_i E_i$. This means that in the non-ageing case, the approximation of the compliance function by the Dirichlet series corresponds to a Kelvin chain with constant properties of individual elements (Fig. 2). The initial conditions for $\epsilon_i^{cr}$ are $\epsilon_i^{cr}(0) = 0$.

Using (3), (4) and (5), we finally obtain the rate form

$$\dot{\sigma} = E \dot{\epsilon}^{\text{tot}} - E \sum_{i=1}^{N} \left( \frac{\sigma}{\tau_i E_i} - \frac{\epsilon_i^{cr}}{\tau_i} \right).$$

(6)

A generalization of this uniaxial stress–strain constitutive relation is performed under the simplifying assumption that the Poisson ratio for creep is approximately time-independent and about the same as the elastic Poisson ratio $\nu$.

Thus, in view of the assumed isotropy and based on the principle of superposition, the following relation holds:

$$\epsilon^{\text{tot}}(t) = J(t - t') E C^{-1} : \sigma(t') + E C^{-1} : \int_{t'}^{t} J(t, \tau) d\sigma(\tau),$$

(7)

where $\sigma(t)$ is the stress tensor at time $t$, $\epsilon^{\text{tot}}(t)$ is the tensor of engineering strain, and

$$C^{-1}_{klmn} = \frac{-v}{E} \delta_{kl} \delta_{mn} + \frac{1 + v}{2E} \left( \delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm} \right)$$

(8)

is the elastic compliance fourth order tensor (for a review see Bazant, 1993) and $\delta$ is the usual Kronecker symbol. The approximation of the compliance function $J$ from the Dirichlet series (Eq. (2)) leads to the following incremental form

$$\dot{\epsilon} = C : \dot{\epsilon}^{\text{tot}} + C : \sum_{i=1}^{N} \frac{\epsilon_i^{cr}}{\tau_i} - E \sum_{i=1}^{N} \frac{\sigma}{\tau_i E_i},$$

(9)

where $\epsilon_i^{cr}$ is the creep strain tensor related to the $i$th unit of the Dirichlet series in 3D.
2.2. Non-local damage model

We will use in the following the scalar continuum damage model proposed by Pijaudier-Cabot and Bazant (1987). The description of the model is that of Pijaudier-Cabot and Benallal (1993). Despite its simplicity (namely the fact that the model response is the same in tension and compression), this model bears the essential characteristics pertaining to a non-local integral model. The constitutive relation reads:

$$\sigma = (1 - D) C : \epsilon_{el,d},$$

where $D$ is the damage variable and $\epsilon_{el,d}$ is the (elastic) strain tensor.

The growth of damage is defined by a loading function $f$

$$f(\bar{y}, D) = \int_0^{\bar{y}} F(z) dz - D,$$

where $F$ is a function deduced from experimental data and $\bar{y}(x)$ is the average energy release rate due to damage at point $x$ of the solid

$$\bar{y}(x) = \int_V \psi(x - s) y(s) dV.$$  \hspace{1cm} (12)

$V$ is the volume of the structure, $\psi(x - s)$ is a normalized weighting function and $y(s)$ is the energy release rate due to damage at point $s$ defined by

$$y(s) = \frac{1}{2} \epsilon_{el,d}(s) : C : \epsilon_{el,d}(s).$$  \hspace{1cm} (13)

The evolution law is prescribed according to the standard format of non-associated irreversible processes:

$$\dot{D} = \phi \frac{\partial g}{\partial \bar{y}},$$  \hspace{1cm} (14)

with the classical Kuhn–Tucker conditions $\phi \geq 0$, $f \leq 0$ and $\phi f = 0$. The dot over a variable indicates differentiation with respect to time, $g$ is the evolution potential controlling the growth of damage and $\phi$ is the damage multiplier. In this paper for the sake of simplicity, we assume $g = \bar{y}$.

The rate form of (10) reads

$$\dot{\sigma} = (1 - D) C : \dot{\epsilon}_{cr} - \frac{\dot{D}}{1 - D} \epsilon_{el,d}.$$  \hspace{1cm} (15)

2.3. Rate-type creep-damage law

The proposed rate-type formulation of a viscoelastic damageable non-ageing continuum assumes a deteriorating Young’s modulus $E$. In fact, the effective stress $\sigma'$ is defined first according to:

$$\sigma' = \frac{\sigma}{1 - D}.$$  \hspace{1cm} (16)

Then, it is assumed that the relationship between the effective stress and the total strain follows a linear viscoelastic model. This is a rather classical approach, inspired from Lemaitre and Chaboche (1985), which turns out to induce a decrease of the Young’s modulus, only in the constitutive relation (4) in our case, which does not affect the characteristic times in the Kelvin chain. By first substituting into relation (10) the expression of $\epsilon_{el,d}$ given in (3), secondly by deriving with respect to time the new relation and, finally, substituting $\dot{\epsilon}_{cr}$ according to the tensorial version of the relation (5), one obtains the following stress–strain relation

$$\dot{\sigma} = (1 - D) C : \dot{\epsilon}^{tot} - \dot{D} C : \epsilon_{el,d} + (1 - D) C : \sum_{i=1}^N \frac{\epsilon_{cr} i}{\tau_i} - (1 - D) C : \sum_{i=1}^N \frac{\sigma}{\tau_i E_i}.$$  \hspace{1cm} (17)
The present model is also based on the assumption that only a fraction \(0 \leq \alpha \leq 1\) of the energy release rate due to creep contributes to the damage evolution with time. The motivation for introducing this parameter into the constitutive law is that for low stress levels although creep strain can be large (even larger than that corresponding to peak stress for short term loading) there is no significant damage (Mazzotti and Savoia, 2001). Hence, we assume that the function \(y(s)\) defined in (13) is now equal to

\[
y(s) = \frac{1}{2} \left( \sigma' (s) : e^{el,d} (s) + \alpha \sigma' (s) : \sum_{i=1}^{N} e^{cr}_i (s) \right),
\]

or equivalently:

\[
y(s) = \frac{1}{2} \left( e^{el,d} (s) : C : e^{el,d} (s) + \alpha e^{el,d} (s) : C : \sum_{i=1}^{N} e^{cr}_i (s) \right),
\]

where the creep strain is given by the tensorial counterpart of (3).

Consider now an initial state of equilibrium at time \(t_0\) denoted by the state variables \(e^{el,d}_0\) and \(D_0\). The rate constitutive relation describing the behavior of the material from this state is

\[
\dot{\varepsilon} = (1 - D_0) C : \ddot{\varepsilon}^{tot} - D C : e^{el,d}_0 + (1 - D_0) C : \sum_{i=1}^{N} \frac{e^{cr}_i}{\tau_i} - (1 - D_0) E \sum_{i=1}^{N} \frac{e^{cr}_i}{\tau_i E_i},
\]

at the points of the solid where damage grows and

\[
\dot{\varepsilon} = (1 - D_0) C : \ddot{\varepsilon}^{tot},
\]

elsewhere.

3. Strain localization analysis

3.1. Equations of motion

The equations of motion are a set of non-linear integro-differential equations since the constitutive relations themselves are non-linear. Upon linearization of the equation of motion about the initial state \((e^{tot}_0, e^{el,d}_0, D_0)\), the momentum equations become

\[
\text{div} \dot{\varepsilon}(x) = \frac{\partial^2 \mathbf{v}}{\partial t^2},
\]

where \(\mathbf{v}\) is the time derivative of the perturbation applied to the initial state. Eqs. (22) is still non-linear due to the constitutive relations. Linearization is now performed under the assumption \(\dot{f} = 0\). This assumption is classical in the analyses of localization. The solid that follows such a constitutive relation is called linear comparison solid (Hill, 1959). Under this assumption and using the damage law (14) and Eq. (19), the equations of motion become

\[
\text{div} \left( (1 - D_0) C : \ddot{\varepsilon}^{tot} - C : e^{el,d}_0 F(\bar{y}_0) \int V \left( \frac{1}{2} \psi (s) \left( 2 e^{el,d}_0 (x + s) : C : \dot{\varepsilon}^{el,d} (x + s) \right) + \alpha \left( e^{el,d}_0 (x + s) : C : \sum_{i=0}^{M} \dot{\varepsilon}^{cr}_i (x + s) + \sum_{i=0}^{M} e^{cr}_i (x + s) : C : \dot{\varepsilon}^{el,d} (x + s) \right) \right) dV \right) + (1 - D_0) C : \sum_{i=1}^{N} \frac{e^{cr}_i}{\tau_i} - (1 - D_0) E \sum_{i=1}^{N} \frac{e^{cr}_i}{\tau_i E_i} = \rho \frac{\partial^2 \mathbf{v}}{\partial t^2}.\]

The initial state of deformation and damage is assumed homogeneous throughout out the solid of volume \(V\). The volume of the solid is assumed to be large enough so that boundary layer effects introduced by spatial averaging can be neglected.
Let us now consider the propagation of a harmonic wave in the direction defined by $\mathbf{n}$

$$\mathbf{v} = A e^{-i\xi(n \cdot x - ct)},$$

(24)

where $\xi$ is the wave number, $c$ is the phase velocity, $A$ is the amplitude of the perturbation and $i$ is the imaginary constant such that $i^2 = -1$. The resulting rate of deformation is

$$\dot{\varepsilon}_{\text{tot}} = -\frac{1}{2}i\xi(A \otimes \mathbf{n} + \mathbf{n} \otimes A)e^{-i\xi(n \cdot x - ct)},$$

(25)

where $\otimes$ denotes the tensorial product.

This harmonic perturbation is admissible if it satisfies the rate equation of equilibrium. Under the assumption that the state variables $\varepsilon_{\text{el},d}^0$ and $D_0$ are constant throughout the solid, taking into account (7) and noticing that $\dot{\varepsilon}_{\text{cr},i}$ defined by

$$\dot{\varepsilon}_{\text{cr},i} = \sigma_0 \tau_i E_i - \varepsilon_{\text{cr},i,0} \tau_i$$

(26)

depends on the initial state only and is homogeneous over the solid, the following relation is obtained:

$$\left( \mathbf{n} \cdot (1 - D_0) \mathbb{C} \cdot \mathbf{n} - \bar{\psi}(\xi \mathbf{n}) F(\bar{\gamma}_0) \left( \mathbf{n} \cdot \frac{1}{2}(2 - \alpha) \mathbb{C} : \varepsilon_{\text{el},d}^0 \otimes \varepsilon_{\text{el},d}^0 : \mathbb{C} \cdot \mathbf{n} + \mathbf{n} \cdot \frac{1}{2} \alpha \mathbb{C} : \varepsilon_{\text{el},d}^0 \otimes \varepsilon_{\text{tot}}^0 : \mathbb{C} \cdot \mathbf{n} \right) \right) \cdot A$$

$$= \rho c^2 A,$$

(27)

where $\bar{\psi}(\xi \mathbf{n})$ is defined by

$$\bar{\psi}(\xi \mathbf{n}) = \int \psi(s) e^{-i\xi \mathbf{n} \cdot s} dV.$$

(28)

In order to obtain (27), one must substitute (24), (25) and (26) into (23). Since the solid is assumed large, $\bar{\psi}(\xi \mathbf{n})$ reduces to the Fourier transform of the weighting function. By considering the weighting function as isotropic, $\bar{\psi}(\xi \mathbf{n})$ becomes independent of the direction $\mathbf{n}$, hence $\bar{\psi}(\xi \mathbf{n}) \equiv \bar{\psi}(\xi)$. Eq. (27) can be written equivalently as an eigenvalue problem

$$\left[ \mathbf{n} \cdot \mathbb{H}^* \cdot \mathbf{n} - \rho c^2 \mathbb{I} \right] \cdot A = 0,$$

(29)

where $\mathbb{I}$ is the second order identity tensor and

$$\mathbf{n} \cdot \mathbb{H}^* \cdot \mathbf{n} = \mathbf{n} \cdot (1 - D_0) \mathbb{C} \cdot \mathbf{n} - \bar{\psi}(\xi \mathbf{n}) F(\bar{\gamma}_0) \left( \mathbf{n} \cdot \frac{1}{2}(2 - \alpha) \mathbb{C} : \varepsilon_{\text{el},d}^0 \otimes \varepsilon_{\text{el},d}^0 : \mathbb{C} \cdot \mathbf{n} + \mathbf{n} \cdot \frac{1}{2} \alpha \mathbb{C} : \varepsilon_{\text{el},d}^0 \otimes \varepsilon_{\text{tot}}^0 : \mathbb{C} \cdot \mathbf{n} \right).$$

(30)

Eq. (27) admits non-trivial solutions if and only if

$$\det[\mathbf{n} \cdot \mathbb{H}^* \cdot \mathbf{n} - \rho c^2 \mathbb{I}] = 0.$$

(31)

### 3.2 Statics – solutions at the bifurcation point

Condition

$$\det[\mathbf{n} \cdot \mathbb{H}^* \cdot \mathbf{n}] = 0,$$

(32)

is similar to the localization condition in a non-local continuum derived by Pijaudier-Cabot and Benallal (1993). Now, we restrict our analysis for the sake of simplicity to the uniaxial case or to loading paths such that

$$\varepsilon_{\text{el},d}^0 = \kappa_0 \varepsilon_{\text{tot}}^0,$$

(33)

where $\kappa_0$ is a real such that $0 < \kappa_0 \leq 1$. Such paths are typical of creep under constant stress at a given load level or to monotonic loading (Appendix A). Using (33), we obtain after some algebra

$$\frac{1}{2} \left( (2 - \alpha) \kappa_0^2 + \alpha \kappa_0 \right) \mathbf{n} \cdot \mathbb{C} : \varepsilon_{\text{tot}}^0 \cdot (\mathbf{n} \cdot \mathbb{C} \cdot \mathbf{n})^{-1} \cdot \varepsilon_{\text{tot}}^0 : \mathbb{C} \cdot \mathbf{n} = \frac{(1 - D_0)}{\bar{\psi}(\xi \mathbf{n}) F(\bar{\gamma}_0)}.$$

(34)
Let now,

\[ \psi(x) = \psi_0 \exp\left( -\frac{\|x\|^2}{2l_c^2} \right), \]  

where \( l_c \) is the internal length of the non-local continuum and \( \psi_0 \) is a normalizing factor. The Fourier transform of the above is

\[ \hat{\psi} (\xi \mathbf{n}) = \exp \left( -\frac{\xi^2 l_c^2}{2} \right). \]  

Solving (32) consists in finding the normal \( \mathbf{n} \) and the wavelength \( 2\pi/\xi \) satisfying this equation. A geometrical approach proposed by Benallal (1992) for the analysis of localization phenomena in thermo-elasto-plasticity is used here. We have

\[ (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n})^{-1} = \frac{\| \|}{\mu} - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \mathbf{n} \otimes \mathbf{n}, \]  

\[ \mathbf{C} : \mathbf{e}_0^{tot} \cdot \mathbf{n} = \lambda \text{tr}(\mathbf{e}_0^{tot}) \mathbf{n} + 2\mu \mathbf{e}_0^{tot} \cdot \mathbf{n}, \]  

\[ \mathbf{n} : \mathbf{C} : \mathbf{e}_0^{tot} \cdot \mathbf{n} = \lambda \text{tr}(\mathbf{e}_0^{tot}) \mathbf{n} + 2\mu \mathbf{n} \cdot \mathbf{e}_0^{tot} \cdot \mathbf{n}, \]  

where \( \lambda \) and \( \mu \) are the Lamé constants.

\[ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \]  

The normal vector \( \mathbf{n} \) and the bifurcation criterion maps into a set of ellipses in this plane. The size of the ellipses increases as the wave number \( \xi \) increases. Before localization the smallest possible ellipse contains the largest Mohr circle. Bifurcation will occur for the first time when the smallest possible ellipse is tangent to the largest Mohr circle of deformation (the smallest possible ellipse corresponds to \( \psi(\xi) = 1 \) or equivalently to \( \xi = 0 \)). The normal vector \( \mathbf{n} \) can be calculated geometrically from standard Mohr analysis (see Fig. 3). After the first occurrence of bifurcation (initial state beyond the first occurrence of bifurcation) the ellipse will intersect the largest Mohr circle. In this case there is a finite set of normal vectors \( \mathbf{n} \) such that bifurcation is possible. However, each vector \( \mathbf{n} \) corresponds to an ellipse whose radius is defined by \( \xi \) (Fig. 4). The wavelength corresponding to each vector \( \mathbf{n} \) is unique and it is obtained after (40)

\[ \lambda(n) = \frac{2\pi}{\xi} \]  

\[ = \pi l_c \sqrt{2 \log \left( \frac{4\mu (C^{el,d})^2 + 4\mu^2 \left( e^{el,d} + \frac{\lambda}{2\mu} \text{tr}(e^{el,d}) \right)^2}{(1-D_0)\frac{1}{2}(2-\alpha)\kappa_0^2 + \alpha\kappa_0} F(\bar{y}_0)} \right)}. \]  

This wavelength is related to the width of the localization zone.

The above relation can be written also in terms of the elasto-damage strain

\[ \lambda(n) = \frac{2\pi}{\xi} \]  

\[ = \pi l_c \sqrt{2 \log \left( \frac{4\mu (C^{el,d})^2 + 4\mu^2 \left( e^{el,d} + \frac{\lambda}{2\mu} \text{tr}(e^{el,d}) \right)^2}{(1-D_0)\frac{1}{2}(2-\alpha)\kappa_0^2 + \alpha\kappa_0} F(\bar{y}_0)} \right)}. \]
3.2.1. Creep effect on the width of the localization zone

In order to study the effect of creep on the width of the localization zone, we consider two different loading histories. The state variables will be denoted with the index $sc$ for the first one, and with index $mc$ for the second one, in which the ratio $\kappa_0^c$ of elasto-damage strain to total strain is smaller ($\kappa_0^{sc} > \kappa_0^{mc}$).

We assume that at the bifurcation points of the two different histories, the elasto-damage strain is the same $\varepsilon_0^{(el,d),sc} = \varepsilon_0^{(el,d),mc}$. Since the elasto-damage strain is fixed, the normal $n$ is fixed as well (Appendix B). Thus, considering $\alpha$ constant and from (43), the wavelengths of the localized mode are such that $l_{mc}(\bar{y}_0) < l_{sc}(\bar{y}_0)$, as long as

$$\left(2 - \alpha + \frac{\alpha}{\kappa_0^{mc}}\right) \frac{F(\kappa_0^{mc})}{1 - D_0(\kappa_0^{mc})} > \left(2 - \alpha + \frac{\alpha}{\kappa_0^{sc}}\right) \frac{F(\kappa_0^{sc})}{1 - D_0(\kappa_0^{sc})},$$

or equivalently as long as $(2 - \alpha + \frac{\alpha}{\kappa_0}) \frac{F(\kappa_0)}{1 - D_0(\kappa_0)}$ is a decreasing function of $\kappa_0$. $(2 - \alpha + \frac{\alpha}{\kappa_0}) \frac{F(\kappa_0)}{1 - D_0(\kappa_0)}$ is a decreasing function of $\kappa_0$ if and only if

$$\frac{\alpha}{\kappa_0^2} \frac{F(\kappa_0)}{1 - D_0(\kappa_0)} \geq \left(2 - \alpha + \frac{\alpha}{\kappa_0}\right) \frac{dF(\bar{y}_0)/d\bar{y}_0(1 - D_0(\bar{y}_0)) + F^2(\bar{y}_0) d\bar{y}_0}{(1 - D_0(\bar{y}_0))^2} \frac{d\bar{y}_0}{d\kappa_0}.$$

(45)

**Remark.** On account of (12), (19) and (33), a sufficient condition for (45) to hold true is

$$\frac{d}{d\bar{y}_0} \left(\frac{1}{F}\right) \leq 1.$$
Thus if the creep effect (creep strains) increases, the critical wavelength of the localized mode decreases whenever (45) holds true. This produces a decrease of the size of the fracture process zone, i.e. an increase of brittleness.

3.2.2. Role of \( \alpha \) on the width of the localization zone

The role of \( \alpha \), defined in Eq. (19) is quite straightforward. If we replace \( \alpha \) in (43) by \( \alpha' \) such that \( \alpha' \leq \alpha \), then we conclude that the width of the localization zone increases. In other words, if the contribution of the creep strain in the growth of damage decreases, the wavelength of the localized mode increases. This produces an increase of the size of the fracture process zone and an increase of ductility. For geometrically similar specimens, this increase of ductility is materialized by a shift of the experimental data point towards a strength criterion, to the left.

When damage depends solely upon the elastic energy \( (\alpha = 0) \), the bifurcation criterion becomes a function of only the elastic strain and creep effects can no longer be reproduced.

4. Numerical example

In order to illustrate the above analysis, consider a uniaxial initial state of deformation

\[
\varepsilon_{\text{tot}}^{n} = \begin{bmatrix} \varepsilon_{n1}^{\text{tot}} & 0 & 0 \\ 0 & -\nu\varepsilon_{n2}^{\text{tot}} & 0 \\ 0 & 0 & -\nu\varepsilon_{n3}^{\text{tot}} \end{bmatrix}.
\]

First we will show that the direction of the critical normal \( \mathbf{n} \) to the localization band does not depend, in the uniaxial case, upon the state of strain or on the damage or on the yield function of damage but only upon the elastic parameters of the material. The critical wavelengths can be obtained when the ellipse corresponding to the above initial state of deformation is tangent to the largest Mohr circle leading to the localization criterion

\[
\frac{E(1+\nu)}{2}\left(\varepsilon_{n1}^{\text{tot}}\sin(2\theta)\right)^2 + \frac{E(1-2\nu)(1+\nu)}{4(1-\nu)}\left(1+\cos(2\theta)\right)^2\left(\varepsilon_{n1}^{\text{tot}}\right)^2
- \frac{1}{2}\left(2-\alpha\right)\kappa_0^2 + \alpha\kappa_0\right)\bar{F}(\bar{y}_0)\bar{\psi}(\bar{\xi}) = 0,
\]

where the angle \( \theta \) defines the normal \( \mathbf{n} = (\cos \theta, 0, \sin \theta) \) to the localization band. This angle satisfies

\[
\tan^2(2\theta) = \frac{2\frac{E(1+\nu)}{1-2\nu}((2-\alpha)\kappa_0^2 + \alpha\kappa_0)\bar{F}(\bar{y}_0)\bar{\psi}(\bar{\xi})(\varepsilon_{n1}^{\text{tot}}\right)^2 - (1 - D_0)}{(1 - D_0) - E(1+\nu)((2-\alpha)\kappa_0^2 + \alpha\kappa_0)\bar{F}(\bar{y}_0)\bar{\psi}(\bar{\xi})(\varepsilon_{n1}^{\text{tot}}\right)^2},
\]

as it can be proved by Mohr analysis. Solving (48) for \( \bar{\psi}(\bar{\xi}) \) and substituting in (47) yields

\[
\frac{E(1-2\nu)(1+\nu)}{4(1-\nu)}\left(1+\cos(2\theta)\right)^2 - \frac{4\frac{E(1+\nu)}{1-2\nu}}{\cos^2(2\theta)} + \frac{5E(1+\nu)}{2}\sin^2(2\theta) = 0.
\]

From the above equation, it is now obvious that the direction of the critical normal \( \mathbf{n} \) to the localization depends solely upon the elastic parameters of the material.

Let us now particularize function \( F \) as

\[
F(\bar{y}) = \frac{b_1 + 2b_2(\bar{y} - y^0)}{(1 + b_1(\bar{y} - y^0) + b_2(\bar{y} - y^0)^2)^2}.
\]

For this function \( F \), damage reads as

\[
D(\bar{y}) = 1 - \frac{1}{1 + b_1(\bar{y} - y^0) + b_2(\bar{y} - y^0)^2}.
\]

The above experimentally determined function \( F \) will serve for the numerical results given below where the numerical values of the model parameters are \( E = 32.000 \text{ MPa}, \nu = 0.2, b_1 = 605 \text{ MPa}^{-1}, b_2 = 5.42 \times 10^4 \text{ MPa}^{-2} \) and \( y^0 = 6 \times 10^{-5} \text{ MPa} \). For these model parameters \( \theta \simeq 25.92^\circ \). Note also that as we have an infinite body

\[
\bar{y} = y = \frac{1}{2}\left(1 - \alpha + \frac{\alpha}{k_0}\right)E(e^{el}e^{tot})^2 = \frac{1}{2}\left(1 - \alpha\right)\kappa_0^2 + \alpha\kappa_0 \right)E(e^{tot})^2.
\]
The first occurrence of bifurcation is obtained when \( \psi(\xi) = 1 \). In order to find the possible values of damage \( D_0 \) at the first occurrence of bifurcation, it is useful to write (47) for \( \psi(\xi) = 1 \) as
\[
\left( \frac{1 + \nu}{2} \right) \sin(2\theta) + \frac{(1 - 2\nu)(1 + \nu)(1 + \cos(2\theta))^2}{4(1 - \nu)} \left( \frac{F(\tilde{y}_0)}{(1 - D_0)} \right) \tilde{y}_0 = 1 - \frac{1}{2 - \alpha + \alpha/\kappa}. \tag{52}
\]
The right-hand side of the above equation takes values in between 0 and 1. The behavior of the left-hand side of the above equation is presented in Fig. (5). We see that for every pair of \((\kappa_0, \alpha)\), that is for every value of the right-hand side of (52) in between 0.5 and 1, Eq. (52) can be solved for \( \tilde{y}_0 \). Thus at the first occurrence of bifurcation, \( 5.11 \times 10^{-4} \leq \tilde{y}_0 \leq 1.287 \times 10^{-3} \) depending on the values of \( \kappa_0 \) and \( \alpha \) that yields \( 0.2211 \leq D_0 \leq 0.4517 \). \( D_0 = 0.4517 \) corresponds to the first occurrence of bifurcation in the time-independent case. That means that in the viscoelastic case the initial state of damage \( D_0 \) at the first occurrence of bifurcation is less or equal to that corresponding to the time-independent case.

Next, we examine the validity of criterion (45). We see in Fig. 6 that this criterion is not always fulfilled. Thus the critical wavelengths of the proposed model for fixed initial elasto-damage strain \( \epsilon_{03}^{el}, \epsilon_{02}^{el} = -\nu \epsilon_{01}^{el} \) (uniaxial tension) may decrease or increase as the creep strains increase, i.e. as the initial state of damage \( D_0 \) increases (Fig. 7).

It is again useful to write criterion (45) in a different form
\[
\frac{dF(\tilde{y}_0)(1 - D_0(\tilde{y}_0))}{d\tilde{y}_0} + \frac{F^2(\tilde{y}_0)}{\tilde{y}_0} \geq 1 + \frac{1}{(1 - \alpha + \alpha/\kappa)}. \tag{53}
\]
The left-hand side of the above equation is plotted in Fig. 8 where one can see that as long as \( \tilde{y}_0 \leq 2.98 \times 10^{-3} \) the criterion is fulfilled, since for those values of \( \tilde{y}_0 \) the left-hand side of (53) is greater than 2, that is greater than the right-hand side that takes values in between 1 and 2. The damage corresponding to \( \tilde{y}_0 = 2.98 \times 10^{-3} \) is \( D_0 = 0.6903 \).

From the above analysis, it is deduced that at least for 0.4517 \( \leq D_0 \leq 0.6903 \) and for all \( 0 < \kappa_0 \leq 1 \) and \( 0 \leq \alpha \leq 1 \) the critical wavelengths decrease with increasing creep effect (creep strains) (Fig. 9). This interval given for the initial state of damage \( D_0 \) in order for the necessary condition (45) to hold true is the most conservative one. For \( \kappa_0 \rightarrow 0 \) it expands to \( 0.2211 \leq D_0 \leq 1 \).

Moreover, in Fig. 10 the role of the parameter \( \alpha \) is shown; when the parameter \( \alpha \) decreases the normalized admissible wavelengths increase. In Fig. 10 the admissible wavelengths are plotted against the initial state of total strain \( \epsilon_{03}^{tot} = \epsilon_{02}^{tot} = -\nu \epsilon_{01}^{tot} = -0.00032\nu \) because against the initial damage \( D_0 \) the difference is negligible.

It is also of interest to obtain the smallest possible wavelength \( 2\pi/\xi \). Consider an initial state of deformation given by
\[
\epsilon_0^{tot} = \begin{bmatrix} \epsilon_{01}^{tot} & 0 & 0 \\ 0 & \epsilon_{02}^{tot} & 0 \\ 0 & 0 & \epsilon_{03}^{tot} \end{bmatrix},
\]
Fig. 6. Uniaxial tension: $Q$ as a function of the $\kappa_0$ and elasto-damage strain $\epsilon_{el,d}^{03} = \epsilon_{el,d}^{02} = -\nu \epsilon_{el,d}^{01}$ and for $\alpha = 0.2$. $Q = 2 \frac{F(\kappa_0)}{1 - \frac{D_0}{\kappa_0}} + (2 - \alpha + \frac{\alpha}{\kappa_0}) \frac{dF(\bar{y}_0)/d(1 - D_0(\bar{y}_0)) + F^2(\bar{y}_0)}{2 - D_0(\bar{y}_0)} E(\epsilon_{el}^{01})^2 \geq 0$ is equivalent to (53). The thick black line corresponds to contour $Q = 0$.

Fig. 7. Uniaxial tension: normalized critical wavelengths of the localization mode for $\alpha = 0.2$. Left: as a function of the initial state of damage $D_0$ and elasto-damage strain $\epsilon_{el,d}^{01} = \epsilon_{el,d}^{02} = -\nu \epsilon_{el,d}^{01}$; top right: as a function of the initial state of damage $D_0$ for fixed initial state of elasto-damage strain $\epsilon_{el,d}^{01} = 0.00032$. Shift on the size-effect plot towards LEFM with increasing $D_0$; bottom right: as a function of the initial state of damage $D_0$ for fixed initial state of elasto-damage strain $\epsilon_{el,d}^{01} = 0.00065$. Shift on the size-effect plot towards the strength criterion with increasing $D_0$.

with $\epsilon_{el,d}^{01} \geq \epsilon_{el,d}^{02} \geq \epsilon_{el,d}^{03}$ and $\kappa_0$. The smallest possible wavelength, for a given initial state, corresponds to the ellipse which is tangent to the largest Mohr circle since smaller values of $2\pi/\xi$ correspond to larger ellipses that do not intersect with the admissible region of Mohr plane. Thus this wavelength will be given by
\[ Q_2 = \frac{F(y_0)(1-D_0(y_0))}{dF(y_0)/d(y_0)} = 0, \] versus \( \tilde{y}_0 = y_0. \)\(^\star\)\(^\star\) denotes the point at which the necessary condition (53) fails for the first time. This point corresponds to the case \( \kappa_0 = 1. \)

\[ l = \frac{2\pi}{\xi} = \frac{2\pi l_c}{2\log (2((2-\alpha)k_0^2 + \alpha\kappa_0)(3\lambda + 2\mu)/(3\lambda + \mu))} \]
Due to the variations of the elastic properties, the smallest critical wavelength reads

$$1_{\text{min}} = \frac{\pi l_c}{\sqrt{\log \left( \max_{0<\alpha \leq 1} (2 - \alpha) \kappa_0^2 + \alpha \kappa_0) \right)}} = \frac{\pi l_c}{\sqrt{\log 2}},$$

and equals the smallest possible wavelength of the localization modes of a time-independent case (Pijaudier-Cabot and Benallal, 1993). Thus the localization limiting properties of the non-local model are preserved, that is the minimum wavelength of the localization modes cannot be zero.

5. Conclusion

The localization properties of a rate-dependent material described by a non-local damage-based constitutive relation coupled to linear viscoelasticity has been investigated. The loading paths considered are those for which changes in the loading state occur only under time-independent processes (negligible creep strain) and the uniaxial loading. The localization condition has the same form as that of the rate-independent underlying damage model.

The proposed coupled model preserves localization limiting properties, same as in the rate independent case (Pijaudier-Cabot and Benallal, 1993) and it is consistent with the effect of creep observed experimentally when a certain condition on the growth of the yield function of damage and the initial state of deformation is met. Namely, the minimum wavelength of the localization modes cannot be zero in accordance with energy considerations and the admissible wavelengths decrease when the creep effect (strains) increases under the aforementioned condition. This last property should induce a variation on size effect data for geometically similar specimens: a shift in the size effect plot towards increasing brittleness for an increasing creep strain. For a specific yield function of concrete (used for illustrative purposes) and for uniaxial loading, the condition which yields increasing brittleness with increasing creep effect is fulfilled for initial damage state in a region near the first occurrence of bifurcation. Moreover, when the fraction of creep strain which contributes to damage is decreasing, a shift on the size effect plot occurs towards the left-hand side, i.e. towards the strength criterion and an increase of ductility is expected. The model fails to predict the influence of creep on the wavelength of localized modes when damage is considered to depend solely upon the elastic energy.

Such analytical results are expected to be recovered in computational failure analyses. They are expected to have important practical consequences: as creep develops, the residual capacity of a given structure may decrease; the structure is becoming more brittle, safety margins may be reduced. This of course should hold for severely loaded concrete structures, where damage interacts with creep. It may typically occur in some regions of prestressed concrete structures subjected to stress concentrations.
Appendix A

Consider a time-independent process until the stress tensor takes the value $\epsilon_{\text{ind}}$ and then the loading state remains fixed for the time interval $[t_{\text{ind}}, t_0]$, where $t_0$ is the time at which bifurcation occurs. For this loading path relations (5) would give

$$
epsilon_{\text{el}} = \sum_{i=1}^{N} \epsilon_{\text{el},i} = \sum_{i=1}^{N} \frac{1 - e^{-t/\tau_i}}{E_i} \epsilon_{\text{ind}}. \quad (58)$$

Since

$$
epsilon_{\text{el,d}} = \frac{1}{(1 - D_0)} \epsilon_{\text{ind}}, \quad (59)$$

one may conclude that (33) holds true for

$$\kappa_0 = \frac{1/((1 - D_0)E)}{1/((1 - D_0)E) + \sum_{i=1}^{N} (1 - e^{-t/\tau_i})/E_i}. \quad (60)$$

In the same spirit, relation (33) can be proved for more general loading paths for which the loading state is either fixed or changes through a time-independent process.

In the uniaxial loading now, at the onset of bifurcation, due to the fact that only $\sigma_{01} \neq 0$, relation (7) reads

$$\epsilon_{01} = E \epsilon_{\text{ind}} (a + b) : \sigma_0, \quad (61)$$

where $a$ and $b$ are numbers such that

$$a \epsilon_0 = J(t_0 - t') \sigma(t')$$

and

$$b \epsilon_0 = \int_{t'}^{t_0} J(t - \tau) d\sigma(\tau).$$

Moreover,

$$\epsilon_{\text{el,d}} = \frac{1}{1 - D_0} \epsilon_{\text{ind}}, \quad (62)$$

holds true. Using now (61) and (62), we deduce (33) with $\kappa_0 = 1/((1 - D_0)E(a + b))$.

Appendix B

The normal $n = (\cos \theta, 0, \sin \theta)$ to the localization band can be obtained when the ellipse corresponding to the initial state of strain $\epsilon_0^{tot}$ is tangent to the largest Mohr circle leading to system

$$
\tan^2(2\theta) = \frac{2(\lambda + 2\mu)((2 - \alpha)\kappa_0^2 + \alpha \kappa_0)F(\tilde{\gamma}_0)\tilde{\psi} (\xi)(\epsilon_{01}^{tot} - \epsilon_{03}^{tot})^2 - (1 - D_0)}{(1 - D_0) - 2\mu((2 - \alpha)\kappa_0^2 + \alpha \kappa_0)F(\tilde{\gamma}_0)\tilde{\psi} (\xi)(\epsilon_{01}^{tot} - \epsilon_{03}^{tot})^2}, \quad (63)
$$

$$
\frac{4\mu^2}{\lambda + 2\mu} \left( \frac{\epsilon_{01}^{tot} + \epsilon_{03}^{tot}}{2} + \frac{\epsilon_{01}^{tot} - \epsilon_{03}^{tot}}{2} \cos(2\theta) + \frac{\lambda}{2\mu} \text{tr}(\epsilon_{01}^{tot}) \right)^2 + 4\mu \left( \frac{\epsilon_{01}^{tot} - \epsilon_{03}^{tot}}{2} \sin(2\theta) \right)^2 = \frac{1}{2}((2 - \alpha)\kappa_0^2 + \alpha \kappa_0)F(\tilde{\gamma}_0). \quad (64)
$$

Solving (63) for $\tilde{\psi} (\xi)$ and substituting into (64) yields

$$
\frac{4\mu^2}{\lambda + 2\mu} \left( \frac{\epsilon_{01}^{tot} + \epsilon_{03}^{tot}}{2} + \frac{\epsilon_{01}^{tot} - \epsilon_{03}^{tot}}{2} \cos(2\theta) + \frac{\lambda}{2\mu} \text{tr}(\epsilon_{01}^{tot}) \right)^2 + 4\mu \left( \frac{\epsilon_{01}^{tot} - \epsilon_{03}^{tot}}{2} \sin(2\theta) \right)^2 = \frac{4}{1 + \tan^2(2\theta)} \left( \epsilon_{01}^{tot} - \epsilon_{03}^{tot} \right)^2. \quad (65)
$$
If (33) is satisfied, the above equation (65) reads

\[
\frac{4\mu^2}{\lambda + 2\mu} \left( \frac{\epsilon_{01}}{\epsilon_{03}} + \frac{\epsilon_{01} - \epsilon_{03}}{2} \cos(2\theta) + \frac{\lambda}{2\mu} \text{tr}(\epsilon_{el,d}) \right)^2 + \frac{4\mu}{(\epsilon_{01} - \epsilon_{03})^2} \left( \frac{\epsilon_{el,d}}{2} \sin(2\theta) \right)^2 \]

\[
= 4 \frac{\lambda + 2\mu}{1 + \tan^2(\theta)} \left( \epsilon_{el,d} - \epsilon_{03} \right)^2.
\]

(66)

It is then obvious from (66) that, as long as (33) holds true, the normal \( n \) remains fixed for fixed initial elasto-damage strain \( \epsilon_{0,d} \).

References


