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ON THE SYMMETRIES OF INTEGRABLE PARTIAL DIFFERENCE EQUATIONS

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We investigate the Lie point and generalized symmetries of certain nonlinear integrable equations on quad-graphs. Applications of the symmetry methods to such equations in obtaining group invariant solutions, related to discrete versions of the Painlevé differential equations, are also demonstrated.

1. Introduction

The study of Sophus Lie in the late nineteenth century on the unification and extension of various solution methods for ordinary differential equations led him to introduce the notion of continuous groups of symmetry transformations. During the same period Bäcklund investigated possible extensions of Lie contact transformations, introducing an important class of surface transformations in ordinary space. A remarkable feature of Bäcklund transformations is that due to a commutativity property repeated applications can be performed in a purely algebraic manner. This is known in classical geometry as the Bianchi permutability theorem and represents a nonlinear analogue of the superposition principle for linear homogeneous differential equations. The archetype is given by the equation

$$(p-q)\tan\left(\frac{u_{12}-u}{4}\right) = (p+q)\tan\left(\frac{u_2-u_1}{4}\right). \tag{1}$$

It relates a solution u_{12} of the sine-Gordon equation

$$u_{xy} = \sin u\,, \tag{2}$$

with an arbitrary seed solution u and two solutions u_1 and u_2 obtained from u via the Bäcklund transformations specified by the parameters values p and q, respectively.

On the other hand, equation (1) may be interpreted as a partial difference equation. This interpretation is obtained by simply identifying u_1

and u_2, respectively, with the values attained by the dependent variable u when the discrete independent variables n_1 and n_2 change by a unit step.

Nonlinear partial differential equations (PDEs) possessing Bäcklund transformations and their compatible nonlinear partial difference equations (PΔEs) arising from the associated superposition formulae have been the subject of intensive investigations during the past century, leading to the modern theory of integrable systems. Such systems are characterized also by an extremely high degree of symmetry, and as a result, Lie symmetries and their generalizations have proven to be invaluable tools for generating solutions of difference and differential equations.

Symmetry based techniques applied to difference equations have appeared quite recently in the literature starting from different philosophies, see e.g. [1]-[11] and references therein. In the present work, we investigate the symmetries of certain partial difference equations living on elementary quadrilaterals. Our approach to this problem originates in the interplay between integrable quadrilateral equations and their compatible continuous PDEs, as this has been addressed recently in [5], [7]. We find Lie point and generalized symmetries for the celebrated discrete potential KdV equation. We show that appropriate linear combinations of Lie point and generalized symmetries can be used in obtaining group invariant solutions of the lattice equations. In particular we show that certain reductions of this type lead to discrete versions of the Painlevé differential equations.

2. Symmetries of quadrilateral equations

Central to our considerations are integrable discrete equations on quad-graphs, i.e. certain equations associated to planar graphs with elementary quadrilaterals faces. In particular, we consider equations where fields are assigned on the vertices and the lattice parameters on the edges of $\mathbb{Z}^2$. In the simplest case, one has complex fields $f : \mathbb{Z}^2 \to \mathbb{C}$ assigned on the vertices at sites (n_1, n_2) which vary by unit steps only, and complex lattice parameters α_1, α_2 assigned on the edges of an elementary square (Fig. 1). The basic building block of such equations consists of a relation of the form

$$\mathcal{H}(f, f_{(1,0)}, f_{(0,1)}, f_{(1,1)}; \alpha_1, \alpha_2) = 0\,, \tag{3}$$

which relates the values of four fields residing on the four vertices of an elementary quadrilateral. The forward shifted value of a field along n_1 and n_2 will be denoted by $f_{(0,1)}, f_{(0,1)}$ respectively, i.e.

$$f_{(1,0)} = f(n_1 + 1, n_2), \quad f_{(1,1)} = f(n_1 + 1, n_2 + 1)\,. \tag{4}$$

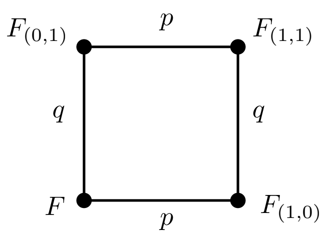

Figure 1. An elementary quadrilateral.

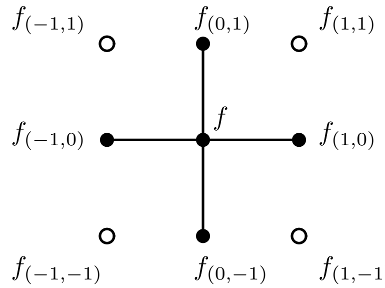

Figure 2. A cross configuration.

A specific equation of the type (3) is given by the equation (1). Its linearized version is the following PΔE

$$(p-q)(f_{(1,1)}-f) = (p+q)(f_{(0,1)}-f_{(1,0)})\,. \tag{5}$$

Let Q be a scalar function which depends on f and its shifted value forming the cross configuration of Fig. 2. We denote the *first prolongation* of a vector field $X_Q = Q\,\partial_f$, by the vector field

$$X_Q^{(1)} = Q\,\partial_f + Q_{(-1,0)}\,\partial_{f_{(-1,0)}} + Q_{(0,1)}\,\partial_{f_{(0,1)}} + Q_{(0,-1)}\,\partial_{f_{(0,-1)}} + Q_{(0,1)}\,\partial_{f_{(0,1)}}\,. \tag{6}$$

Similarly, the second prolongation of X_Q is denoted by

$$\begin{aligned} X_Q^{(2)} = X_Q^{(1)} &+ Q_{(-1,-1)}\partial_{f_{(-1,-1)}} + Q_{(-1,1)}\partial_{f_{(-1,1)}} + \\ &Q_{(1,-1)}\partial_{f_{(1,-1)}} + Q_{(1,1)}\partial_{f_{(1,1)}} + \dots \end{aligned} \tag{7}$$

Definition 2.1. We say that $X_Q = Q\,\partial_f$ is a symmetry generator of the quadrilateral equation (3), if and only if

$$X_Q^{(2)}(\mathcal{H}) = 0\,, \tag{8}$$

holds for all solutions of equation (3). Thus, in equation (8) we should take into account (3), and its consequences.

It may be seen from the above definition that equation (8) is a *linear* functional relation for Q. Solutions of the latter equation provide symmetries admitted by equation (3). An indirect approach for determining solutions of the corresponding functional relation for Q, once a specific equation of the type (3) is given, is to derive first a compatible set of differential-difference and partial differential equations, by interchanging the role of the discrete variables (n_1, n_2) with that of the continuous parameters (p, q). The reasoning behind this construction is that one could set up a natural framework for the description of the symmetries and reductions of discrete systems, by exploiting the notion of Lie-point symmetries and the infinitesimal methods for obtaining them, which are well known for the continuous PDEs. We next illustrate the relevant construction for the PΔE (5).

A particular solution of equation (5) is

$$f=\left(\frac{p-\lambda}{p+\lambda}\right)^{n_1}\left(\frac{q-\lambda}{q+\lambda}\right)^{n_2}, \tag{9}$$

$\lambda\in\mathbb{C}$. Differentiating f with respect to p, (respectively q) and rearranging terms, we easily find that f also satisfies the differential-difference equations (DΔEs)

$$f_p=\frac{n_1}{2p}(f_{(-1,0)}-f_{(1,0)}), \qquad f_q=\frac{n_2}{2q}(f_{(0,-1)}-f_{(0,1)}), \tag{10}$$

where the minus sign denotes backward shift in the direction of the corresponding discrete variable.

By interchanging completely the role of the lattice variables n_1, n_2 with that of the continuous lattice parameters p, q, the aim now is to find a PDE which is compatible with equations (5) and (10). Such a PDE is the fourth order equation obtained from the Euler-Lagrange equation

$$\partial_{pq}\left(\frac{\partial L}{f_{pq}}\right)-\partial_p\left(\frac{\partial L}{f_p}\right)-\partial_q\left(\frac{\partial L}{f_q}\right)=0, \tag{11}$$

for the variational problem associated with the Lagrangian density

$$L=\frac{1}{2}(p^2-q^2)(f_{pq})^2+\frac{2}{p^2-q^2}(n_2f_p-n_1f_q)(n_2p^2f_p-n_1q^2f_q). \tag{12}$$

Two of the divergence symmetries of Lagrangian L are the scaling transformations

$$p\mapsto\alpha p, \quad q\mapsto\alpha q, \quad f\mapsto\beta f, \qquad \alpha,\beta\in\mathbb{C},\ \alpha,\beta\neq 0. \tag{13}$$

Since every divergence symmetry of a variational problem is inherited as a Lie-point symmetry by the associated Euler-Lagrange equations, the transformations (13) are Lie-point symmetries of equations (11). They correspond to the characteristic symmetry generator

$$X_Q=Q\,\partial_f, \quad \text{where} \quad Q=c_1(pf_p+qf_q)+c_2\,f, \qquad c_1, c_2\in\mathbb{C}. \tag{14}$$

In view of the compatible DΔEs (10), the characteristic Q takes the form

$$Q=\frac{c_1}{2}\left(n_1(f_{(-1,0)}-f_{(1,0)})+n_2(f_{(0,-1)}-f_{(0,1)})\right)+c_2f. \tag{15}$$

Equations (5), (10) and (11) form a compatible set of equations, in the sense that they share a non empty set of solutions. By virtue of this fact and since the symmetry generator X_Q given by (14) maps solutions to solutions of PDE (11), X_Q, with Q given by (15), should generate a symmetry of the discrete equation (5). In other words, Q given by (15) should satisfy

$$(p-q)(Q_{(1,1)}-Q)=(p+q)(Q_{(0,1)}-Q_{(1,0)})\,, \tag{16}$$

for all solutions f of (5). Taking into account equation (5) and its backward discrete consequences, we easily find that equation (16) holds. Thus, Q is indeed a symmetry characteristic of equation (5).

The above considerations justify the reason why the symmetry characteristic Q of a general quadrilateral equation (3), depends initially on the values of f assigned on the points which form the *cross* configuration of Fig. 2. In general, this dependence could be arbitrary. Indeed, as it is illustrated in the following sections, symmetries which correspond to extended cross configurations can be found from known ones.

3. Symmetries of equation (5)

The symmetries of equation (5) are determined from the functional equation (16). Two simple solutions of the latter give the symmetry generators

$$X_1=(\,\mu+\lambda(-1)^{n_1+n_2}\,)\partial_f, \qquad X_2=f\partial_f\,. \tag{17}$$

Symmetry characteristics corresponding to the cross configuration of Fig. 2, and which can be found by exploiting the correspondence with the continuous PDE, are given by the vector fields

$$Y_1=\left(f_{(1,0)}-f_{(-1,0)}\right)\partial_f, \qquad Y_2=\left(f_{(0,1)}-f_{(0,-1)}\right)\partial_f, \tag{18}$$

$$Z=\left(n_1(f_{(1,0)}-f_{(-1,0)})+n_2(f_{(0,1)}-f_{(0,-1)})\right)\partial_f\,. \tag{19}$$

The latter serve to construct an infinite number of symmetries. This follows from the fact that the commutator of two symmetry generators is again a symmetry generator. Let

$$Q_{[i,0]}=f_{(i,0)}-f_{(-i,0)}\,, \qquad Q_{[0,j]}=f_{(0,j)}-f_{(0,-j)} \qquad\qquad i,j\in\mathbb{N}\,, \tag{20}$$

be the characteristics of the vector fields

$$Y_{Q_{[i,0]}}=Q_{[i,0]}\partial_f\,, \qquad Y_{Q[0,j]}=Q_{[0,j]}\partial_f\,, \qquad i,j\in\mathbb{N}\,. \tag{21}$$

By induction we find that

$$Y_{Q_{[i-1,0]}} + \frac{1}{i}\,[Z, Y_{Q_{[i,0]}}] = Y_{Q_{[i+1,0]}}\,, \tag{22}$$

$$Y_{Q_{[0,j-1]}} + \frac{1}{j}\,[Z, Y_{Q_{[0,j]}}] = Y_{Q_{[0,j+1]}}\,, \tag{23}$$

holds $\forall\, i, j \in \mathbb{N} \setminus \{0\}$. Repeated applications of the commutation relations (22), (23) produce new symmetries of equation (5), and thus the vector field Z represents a *master symmetry*. The generated new symmetries correspond to extended cross configurations.

4. Symmetries of the discrete potential KdV equation

We next demonstrate how the above considerations can be applied equally well to a nonlinear discrete equation, namely the discrete Korteweg-de Vries (KdV) equation [12]

$$(f_{(1,1)} - f)(f_{(1,0)} - f_{(0,1)}) = p - q\,. \tag{24}$$

Exploiting the symmetries of the continuous compatible PDE and the interplay between the compatible set of differential and difference equations, we have found [13] the following symmetries for the discrete KdV equation

$$X_1 = \partial_f\,, \quad X_2 = (-1)^{n_1+n_2} f \partial_f\,, \qquad X_3 = (-1)^{n_1+n_2} \partial_f\,, \tag{25}$$

$$Y_1 = \frac{1}{f_{(1,0)} - f_{(-1,0)}}\,\partial_f, \quad Y_2 = \frac{1}{f_{(0,1)} - f_{(0,-1)}}\,\partial_f, \tag{26}$$

$$Z_1 = \left(\frac{n_1}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2}{f_{(0,1)} - f_{(0,-1)}}\right)\partial_f\,, \tag{27}$$

$$Z_2 = \left(\frac{n_1\, p}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2\, q}{f_{(0,1)} - f_{(0,-1)}} - \frac{1}{2} f\right)\partial_f\,, \tag{28}$$

Taking the commutator of Z_1 with Y_1, one finds the new symmetry generator

$$[Z_1, Y_1] = \frac{1}{(f_{(1,0)} - f_{(-1,0)})^2}\left(\frac{1}{f - f_{(2,0)}} + \frac{1}{f_{(-2,0)} - f}\right)\partial_f \tag{29}$$

and a similar relation can be found for the commutator $[Z_1, Y_2]$. Further new symmetries are obtained by taking the commutator of Z_1 with the resulting new symmetries.

5. Symmetry reduction on the lattice

Let $\mathcal{H} = 0$ be a quadrilateral equation of the form (3) and $X_Q = Q\partial_f$ a symmetry generator. In analogy with the continuous PDEs, we adopt the following.

Definition 5.1. We say that a solution f of equation $\mathcal{H} = 0$ is an invariant solution under X_Q, if it satisfies in addition to $\mathcal{H} = 0$, the compatible constraint $X(f) = 0$, or equivalently $Q = 0$.

We next demonstrate the notion of invariant solutions of discrete quadrilateral equations by considering a specific symmetry reduction of the discrete KdV (24). For the symmetry constraint we choose a linear combination of Y_1 and Y_2 given by (26). The corresponding invariant solutions are obtained from the compatible system

$$(f_{(1,1)} - f)(f_{(1,0)} - f_{(0,1)}) = p - q\,, \tag{30}$$

$$c(f_{(1,0)} - f_{(-1,0)}) = f_{(0,1)} - f_{(0,-1)} \tag{31}$$

The aim is to eliminate from the above system, one direction of the lattice, i.e. to derive an ordinary difference equation. To this end, we define auxiliary variables

$$x = f_{(1,1)} - f\,, \qquad a = f_{(1,0)} - f_{(-1,0)}\,, \tag{32}$$

$$y = f_{(1,0)} - f_{(0,1)}\,,\ b = f_{(0,1)} - f_{(0,-1)}\,. \tag{33}$$

It follows from equations (32)-(33) that

$$b_{(1,0)} = x - y_{(0,-1)}\,,\ b = x_{(0,-1)} - y\,, \tag{34}$$

$$a_{(0,1)} = x + y_{(-1,0)}\,,\ a = x_{(-1,0)} + y\,. \tag{35}$$

With the help of the auxiliary variables (32)-(33), we arrive at the following OΔE

$$w_{(1,0)} = \frac{\alpha\, w + \beta}{\gamma\, w + \delta}\,, \tag{36}$$

where $w = x\, x_{(-1,0)}$ and the parameters are given by $\alpha = -\delta = r\, c$, $\beta = r^2(1 + c)$, $\gamma = 1 - c$ and $r = p - q$. Equation (36) is a discrete Riccati equation which can be solved explicitly, by using the symmetry generator [1]

$$X = \big(\gamma w^2 + (\delta - \alpha)w - \beta\big)\partial_w\,. \tag{37}$$

It should be noted that, when $c = -1$, the invariant solutions obtained above correspond to the periodic reduction $f_{(-1,1)} = f_{(1,-1)}$.

6. Symmetry reductions to discrete Painlevé equations

In this section we show how discrete versions of the Painlevé equations arise from invariant solutions of the discrete KdV. In particular, here we demonstrate two such symmetry reductions using the symmetry generators

$$W_1 = Z_1 + \lambda X_1, \qquad W_2 = Z_1 + \lambda X_2,$$

respectively, where the symmetry generators X_1, X_2 and Z_1 are given by equations (25) and (27).

Invariant solutions under W_1 are solutions of discrete KdV, subject to the compatible constraint $W_1(f) = 0$, i.e.

$$\frac{n_1}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2}{f_{(0,1)} - f_{(0,-1)}} + \lambda = 0\,, \tag{38}$$

which in terms of the auxiliary variables (32) reads

$$\frac{n_1}{a} + \frac{n_2}{b} + \lambda = 0\,. \tag{39}$$

Using the auxiliary variables (32) and discrete KdV equation we have

$$a = x_{(-1,0)} + \frac{r}{x}\,, \tag{40}$$

$r = p - q$. On the other hand, a straightforward calculation shows that

$$\frac{1}{b_{(1,0)}} = \frac{r}{x^2}\frac{1}{b} + \frac{1}{x}\,. \tag{41}$$

Eliminating the variable b between equations (39), (41) we get

$$-\left(\lambda + \frac{n_1 + 1}{a_{(1,0)}}\right) = -\frac{r}{w^2}\left(\lambda + \frac{n_1}{a}\right) + \frac{n_2}{w}\,. \tag{42}$$

Substituting a given by equation (40) into (42) and rearranging terms we arrive at the second order difference equation

$$\frac{(n_1+1)r}{xx_{(1,0)} + r} + \frac{n_1 r}{xx_{(-1,0)} + r} = n_1 + n_2 + 1 - \lambda\frac{r}{x} + \lambda x\,, \tag{43}$$

known as alternate discrete Painlevé II equation [14].

Invariant solutions of KdV under the symmetry W_2 are obtained from the constraint

$$\frac{n_1}{f\,f_{(1,0)} - f\,f_{(-1,0)}} + \frac{n_2}{f\,f_{(0,1)} - f\,f_{(0,-1)}} + \lambda(-1)^{n_1+n_2} = 0\,. \tag{44}$$

In a similar manner as previously, we arrive at the following coupled system of difference equations

$$(y_{(-1,0)} - x)\,(x - y) = r\,x\,, \tag{45}$$

$$(n_1+1)\frac{x_{(1,0)}-y}{x_{(1,0)}-x}+n_1\frac{x-y_{(-1,0)}}{x-x_{(-1,0)}} = n_1+n_2+1+ \lambda(-1)^{n_1+n_2}(y-y_{(-1,0)})\,, \quad (46)$$

for the auxiliary variables (group invariants under the symmetry X_2)

$$x=f\,f_{(1,0)}\,,\quad y=f_{(1,0)}\,f_{(1,1)}\,. \quad (47)$$

The system of equations (45), (46) can be decoupled for the variable x. Indeed, equation (46) can be integrated once to give

$$n_1\frac{x-y_{(-1,0)}}{x-x_{(-1,0)}}=\frac{n_1+n_2}{2}+\frac{1}{4}-\lambda(-1)^{n_1+n_2}y_{(-1,0)}+\gamma(n_2)(-1)^{n_1}\,. \quad (48)$$

Due to a compatibility condition, it turns out that it is necessary to set $\gamma(n_2)=c(-1)^{n_2}$. Solving equation (48) for $y_{(-1,0)}$ and inserting into equation (45) we obtain a second order difference equation for the variable x, involving the parameters r,λ,n_2,c. The explicit connection of the latter equation with the various discrete Painlevé equations known so far, cf. [15] and references therein, is under investigation.

7. Concluding remarks

The main purpose of this work was to demonstrate that the notions of symmetry and invariance on the discrete level arise naturally from the interplay between PΔEs and PDEs that share a common set of solutions. Moreover, certain symmetries which admit the aforementioned cross configuration can be used to derive invariant solutions, in exact analogy with the invariant solutions of the continuous PDEs. Recently in [16], the discrete multi-field Boussinesq system and the compatible PDEs were investigated. It was shown that scaling invariant solutions of the relevant PDEs are built from solutions of higher Painlevé equations, which potentially lead to solutions in terms of new transcendental functions. We expect that appropriate reductions of the discrete Boussinesq system using the admitted symmetries, will lead to new discrete equations of Painlevé type, [17]. A detailed study of all inequivalent reductions of this kind, for various integrable discrete equations on quad-graphs, will be given elsewhere.

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