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Symmetries and Group Invariant Reductions of Integrable Partial Difference Equations

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The interplay between the symmetries of compatible discrete and continuous integrable systems in two dimensions is investigated. Master and higher symmetries for certain quadrilateral lattice equations are found. The usage of symmetries in obtaining group invariant reductions on the lattice is also discussed.

1 Introduction

The investigations of Bäcklund, in the late nineteenth century, of possible extensions of Lie contact transformations led him to introduce an important class of surface transformations in ordinary space. The intimate connection of Bäcklund transformations with certain type nonlinear equations, which from a modern perspective are called integrable systems, has been the subject of intensive investigations over the past century. A detailed account on Bäcklund transformations can be found in the recent works [1, 2]. Integrable systems are also characterized by an extremely high degree of symmetry. As a result, Lie symmetries and their generalizations have proven to be invaluable tools for generating solutions and obtaining classification results for this kind of systems, cf [3] and contributions in this volume.

Due to a commutativity property, Bäcklund transformations possess the interesting feature that repeated applications can be performed in a purely algebraic fashion. This is known in classical geometry as the Bianchi permutability theorem and represents a nonlinear analogue of the superposition principle for linear homogeneous differential equations. The prototypical example is given by the equation

$$(p-q)\tan\left(\frac{u_{12}-u}{4}\right)=(p+q)\tan\left(\frac{u_2-u_1}{4}\right). \qquad (1)$$

It relates a solution u_{12} of the sine-Gordon equation

$$u_{xy}=\sin u\,, \qquad (2)$$

with an arbitrary seed solution u and two solutions u_1 and u_2 obtained from u via the Bäcklund transformations specified by the parameters values p and q, respectively.

On the other hand, equation (1) may be interpreted as a partial difference equation. This interpretation is obtained by simply identifying u_1 and u_2, respectively, with the values attained by the dependent variable u when the discrete independent variables n_1 and n_2 change by a unit step.

Recent advances in the theory of integrable systems show that discrete systems are equally important to their continuous analogues, and their study has led to new insights into the structures behind the more familiar continuous systems. Thus, standard symmetry techniques applied to integrable discrete equations have attracted the attention of many investigators, see e.g. [4–11]. More general symmetry approaches are being pursued starting from different philosophies, see e.g. [12–18] and references therein.

In the present work, symmetries and invariant reductions of certain partial difference equations on elementary quadrilaterals are investigated. The approach to this problem originates in the interplay between integrable quadrilateral equations and their compatible continuous PDEs, as this has been addressed recently in [8, 11].

2 Symmetries of Quadrilateral Equations

Central to our considerations on the discrete level are equations on quadrilaterals, i.e. equations of the form

$$\mathcal{H}(F_{(0,0)}, F_{(1,0)}, F_{(0,1)}, F_{(1,1)}; p, q) = 0\,. \tag{3}$$

They may be regarded as the discrete analogues of hyperbolic type partial differential equations (PDEs) involving two independent variables. The dependent variables (fields) are assigned on the vertices at sites (n_1, n_2) which vary by unit steps only, and the continuous lattice parameters $p, q \in \mathbb{C}$ are assigned on the edges of an elementary quadrilateral (Fig. 1). The updates of a lattice variable $F \in \mathbb{C}$, along a shift in the n_1 and n_2 direction of the lattice are denoted by $F_{(0,1)}, F_{(0,1)}$ respectively, i.e.

$$F_{(1,0)} = F(n_1\!+\!1, n_2), \quad F_{(0,1)} = F(n_1, n_2\!+\!1), \quad F_{(1,1)} = F(n_1\!+\!1, n_2\!+\!1)\,. \tag{4}$$

A specific equation of the type (3) is given by the Bianchi lattice (1). Its linearized version is the partial difference equation (PΔE)

$$(p-q)(f_{(1,1)} - f) = (p+q)(f_{(0,1)} - f_{(1,0)})\,. \tag{5}$$

The aim now is to find the symmetries of equation (5) and successively to find the corresponding group invariant solutions. An indirect approach in dealing with such a problem is to derive first a compatible set of differential-difference and partial differential equations, by interchanging the role of the discrete variables (n_1, n_2) with that of the continuous parameters (p, q). The reasoning behind this construction is that one could set up a natural framework for the description of

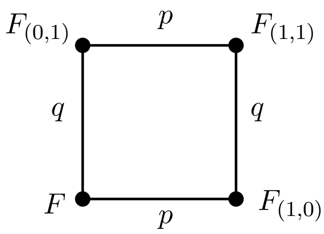

Figure 1. An elementary quadrilateral.

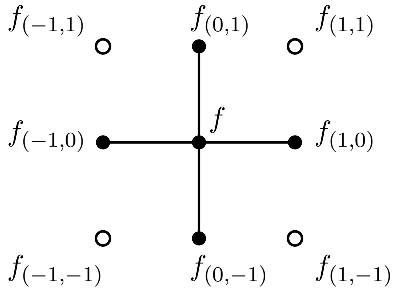

Figure 2. A cross configuration.

the symmetry and the symmetry reduction of discrete systems, by exploiting the notion of Lie-point symmetries and the infinitesimal methods for obtaining them, which are well known for the continuous PDEs. We next illustrate the relevant construction for the PΔE (5).

A particular solution of equation (5) is

$$f = \left(\frac{p-\lambda}{p+\lambda}\right)^{n_1}\left(\frac{q-\lambda}{q+\lambda}\right)^{n_2}, \tag{6}$$

$\lambda \in \mathbb{C}$. Differentiating f with respect to p, (respectively q) and rearranging terms, we easily find that f also satisfies the differential-difference equations (DΔEs)

$$f_p = \frac{n_1}{2p}(f_{(-1,0)} - f_{(1,0)}), \quad f_q = \frac{n_2}{2q}(f_{(0,-1)} - f_{(0,1)}), \tag{7}$$

where the minus sign denotes backward shift in the direction of the corresponding discrete variable.

By interchanging completely the role of the lattice variables n_1, n_2 with that of the continuous lattice parameters p, q, the aim now is to find a PDE which is compatible with equations (5) and (7). Using similar considerations as in [19], we find that such a PDE is the fourth order equation obtained from the Euler-Lagrange equation

$$\partial_{pq}\left(\frac{\partial L}{f_{pq}}\right) - \partial_p\left(\frac{\partial L}{f_p}\right) - \partial_q\left(\frac{\partial L}{f_q}\right) = 0, \tag{8}$$

for the variational problem associated with the Lagrangian density

$$L = \frac{1}{2}(p^2-q^2)(f_{pq})^2 + \frac{2}{p^2-q^2}(n_2 f_p - n_1 f_q)(n_2 p^2 f_p - n_1 q^2 f_q). \tag{9}$$

Two of the divergence symmetries of Lagrangian L are the scaling transformations

$$p \mapsto \alpha p, \quad q \mapsto \alpha q, \quad f \mapsto \beta f, \quad \alpha, \beta \in \mathbb{C}, \ \alpha, \beta \neq 0. \tag{10}$$

Since every divergence symmetry of a variational problem is inherited as a Lie-point symmetry by the associated Euler-Lagrange equations, the transformations (10) are Lie-point symmetries of equations (8). They correspond to the characteristic symmetry generator

$$X_Q = Q\,\partial_f, \quad \text{where} \quad Q = c_1(pf_p + qf_q) + c_2\,f, \quad c_1\,, c_2 \in \mathbb{C}. \tag{11}$$

In view of the compatible DΔEs (7), the characteristic Q takes the form

$$Q = \frac{c_1}{2}\left(n_1(f_{(-1,0)} - f_{(1,0)}) + n_2(f_{(0,-1)} - f_{(0,1)})\right) + c_2 f\,. \tag{12}$$

Equations (5), (7) and (8) form a compatible set of equations, in the sense that they share a common set of solutions. By virtue of this fact and since the symmetry generator X_Q given by (11) maps solutions to solutions of PDE (8), X_Q, with Q given by (12), should generate a symmetry of the discrete equation (5). In other words, Q given by (12) should satisfy

$$(p-q)(Q_{(1,1)} - Q) = (p+q)(Q_{(0,1)} - Q_{(1,0)})\,, \tag{13}$$

for all solutions f of (5). It should be noted that Q depends on the values of f and the four adjacent values on the lattice. Taking into account equation (5) and its backward discrete consequences, we easily find that equation (13) holds. Thus, Q is indeed a symmetry characteristic of equation (5).

The above considerations lead us naturally to assume that the symmetry characteristic Q of a general quadrilateral equation (3) initially depends on the values of f assigned on the points which form the *cross* configuration of Fig. 2. In other words, we are led to adopt the following definition.

Let Q be a scalar function which depends on the values of F and their shifts forming the cross configuration of Fig. 2. We denote the *first prolongation* of a vector field $X_Q = Q\,\partial_F$, by the vector field

$$X_Q^{(1)} = Q\,\partial_F + Q_{(-1,0)}\,\partial_{F_{(-1,0)}} + Q_{(0,1)}\,\partial_{F_{(0,1)}} + Q_{(0,-1)}\,\partial_{F_{(0,-1)}} + Q_{(0,1)}\,\partial_{F_{(0,1)}}. \tag{14}$$

Similarly, the second prolongation of X_Q is denoted by

$$X_Q^{(2)} = X_Q^{(1)} + Q_{(-1,-1)}\partial_{F_{(-1,-1)}} + Q_{(-1,1)}\partial_{F_{(-1,1)}} + Q_{(1,-1)}\partial_{F_{(1,-1)}} + Q_{(1,1)}\partial_{F_{(1,1)}}. \tag{15}$$

We say that $X_Q = Q\,\partial_F$ is a symmetry generator of the quadrilateral equation (3), if and only if

$$X_Q^{(2)}(\mathcal{H}) = 0\,, \tag{16}$$

holds, where equation (3) and its backward discrete consequences should be taken into account.

2.1 Symmetries of the Linearized Bianchi Lattice

The symmetries of equation (5) are determined from the functional equation (13). Two simple solutions of the latter give the symmetry generators

$$X_1 = (\mu + \lambda(-1)^{n_1+n_2})\partial_f, \quad X_2 = f\partial_f\,. \tag{17}$$

Symmetry characteristics corresponding to the cross configuration of Fig. 2, and which can be found by exploiting the correspondence with the continuous PDE, are given by the vector fields

$$Y_1 = \left(f_{(1,0)} - f_{(-1,0)}\right)\partial_f, \quad Y_2 = \left(f_{(0,1)} - f_{(0,-1)}\right)\partial_f, \tag{18}$$

$$Z = \left(n_1(f_{(1,0)} - f_{(-1,0)}) + n_2(f_{(0,1)} - f_{(0,-1)})\right)\partial_f\,. \tag{19}$$

The latter serve to construct an infinite number of symmetries. This follows from the fact that the commutator of two symmetry generators is again a symmetry generator. Let

$$Q_{[i,0]} = f_{(i,0)} - f_{(-i,0)}\,, \quad Q_{[0,j]} = f_{(0,j)} - f_{(0,-j)} \quad i,j \in \mathbb{N}\,, \tag{20}$$

be the characteristics of the vector fields

$$Y_{Q_{[i,0]}} = Q_{[i,0]}\partial_f\,, \quad Y_{Q[0,j]} = Q_{[0,j]}\partial_f\,, \quad i,j \in \mathbb{N}\,. \tag{21}$$

By induction we find that

$$Y_{Q_{[i-1,0]}} + \frac{1}{i}\,[Z, Y_{Q_{[i,0]}}] = Y_{Q_{[i+1,0]}}, \;\; Y_{Q_{[0,j-1]}} + \frac{1}{j}\,[Z, Y_{Q_{[0,j]}}] = Y_{Q_{[0,j+1]}}, \tag{22}$$

holds $\forall\, i,j \in \mathbb{N} \setminus \{0\}$. Repeated applications of the commutation relations (22) produce new symmetries of equation (5), and thus the vector field Z represents a *master symmetry*. The generated new symmetries correspond to extended cross configurations.

2.2 Symmetries of the Discrete Korteweg–de Vries Equation

We next demonstrate how the above considerations can be applied equally well to a nonlinear discrete equation, namely the discrete Korteweg–de Vries (KdV) equation [20]

$$(f_{(1,1)} - f)(f_{(1,0)} - f_{(0,1)}) = p - q\,. \tag{23}$$

Recently in [22], the compatible differential-difference system

$$f_p = \frac{n_1}{f_{(1,0)} - f_{(-1,0)}}\,, \quad f_q = \frac{n_2}{f_{(0,1)} - f_{(0,-1)}}\,, \tag{24}$$

was derived, along with the compatible PDE which is the Euler-Lagrange equation for the variational problem associated with the Lagrangian density

$$L = (p-q)\frac{(f_{pq})^2}{f_p f_q} + \frac{1}{p-q}\left((n_2)^2\frac{f_p}{f_q} + (n_1)^2\frac{f_q}{f_p}\right) . \tag{25}$$

The importance of the above Lagrangian stems from the fact that the commuting generalized symmetries of the associated Euler-Lagrange equation *generate* the complete hierarchy of the KdV soliton equations, (cf [21] for generalizations of the above results). Moreover, the Euler–Lagrange equation acquires a certain physical significance, since it incorporates the hyperbolic Ernst equation for an Einstein–Weyl field [23]. Thus, it would be interesting to find symmetries and special solutions on the discrete level as well.

Exploiting the symmetries of the continuous PDE and the interplay between the compatible set of differential and difference equations, we find the following symmetries of the discrete KdV equation

$$X_1 = \partial_f\,, \quad X_2 = (-1)^{n_1+n_2} f\partial_f\,, \tag{26}$$

$$Y_1 = \frac{1}{f_{(1,0)} - f_{(-1,0)}}\,\partial_f, \quad Y_2 = \frac{1}{f_{(0,1)} - f_{(0,-1)}}\,\partial_f, \tag{27}$$

$$Z_1 = \left(\frac{n_1}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2}{f_{(0,1)} - f_{(0,-1)}}\right)\partial_f\,, \tag{28}$$

$$Z_2 = \left(\frac{n_1\, p}{f_{(1,0)} - f_{(-1,0)}} + \frac{n_2\, q}{f_{(0,1)} - f_{(0,-1)}} - \frac{1}{2}f\right)\partial_f\,, \tag{29}$$

Taking the commutator of Z_1 with Y_1, one finds the new symmetry generator

$$[Z_1, Y_1] = \frac{1}{(f_{(1,0)} - f_{(-1,0)})^2}\left(\frac{1}{f - f_{(2,0)}} + \frac{1}{f_{(-2,0)} - f}\right)\partial_f \tag{30}$$

and a similar relation can be found for the commutator $[Z_1, Y_2]$. Further new symmetries are obtained by taking the commutator of Z_1 with the resulting new symmetries, which are omitted here because of their lenght.

3 Symmetry Reduction on the Lattice

Let $\mathcal{H} = 0$ be a quadrilateral equation of the form (3) and X_Q a symmetry generator. In analogy with the continuous PDEs, we say that a solution of $\mathcal{H} = 0$ is invariant under X_Q, if it satisfies the compatible constraint $Q = 0$.

Let us now consider the linearized Bianchi lattice (5) and a linear combination of the symmetries Y_1 and Y_2 given by equation (18). The corresponding invariant solutions are obtained from the compatible system

$$(p-q)(f_{(1,1)}-f) = (p+q)(f_{(0,1)}-f_{(1,0)})\,, \quad f_{(1,0)}-f_{(-1,0)} = c(f_{(0,1)}-f_{(0,-1)}). \tag{31}$$

The method for obtaining the invariant solutions on the lattice is similar to the direct substitution method for the invariant solutions of PDEs. The aim is to derive from the above discrete system, equations where the variables are given in terms of only one direction of the lattice, i.e. to derive an ordinary difference equation. To this end, we define auxiliary dependent variables

$$x = f_{(1,1)} - f\,, \qquad a = f_{(1,0)} - f_{(-1,0)}\,, \tag{32}$$

$$y = f_{(1,0)} - f_{(0,1)}\,, \quad b = f_{(0,1)} - f_{(0,-1)}\,. \tag{33}$$

It follows from equations (32)–(33) that

$$b_{(1,0)} = x - y_{(0,-1)}\,, \quad b = x_{(0,-1)} - y\,, \tag{34}$$

$$a_{(0,1)} = x + y_{(-1,0)}\,, \quad a = x_{(-1,0)} + y\,. \tag{35}$$

Using the above relations and the system (31), we arrive at the second order linear ordinary difference equation (OΔE) for the variable x

$$x_{(2,0)} - \big(c\,(r - r^{-1}) - (r + r^{-1})\big)\,x_{(1,0)} + x = 0\,, \tag{36}$$

where $r = (q+p)/(q-p)$. Equation (36) can be easily solved, giving

$$x = c_1(n_2)\,{\mu_1}^{n_1} + c_2(n_2)\,{\mu_2}^{n_2}\,, \tag{37}$$

where μ_1, μ_2 are the two roots of the characteristic polynomial of equation (36). In a similar manner, the arbitrary functions c_1, c_2 of n_2 are determined from (31), (32)–(33) and their consequences, leading finally to the invariant form of f.

We conclude this section by considering a specific symmetry reduction of the discrete KdV (23). For the compatible symmetry constraint we choose a linear combination of Y_1 and Y_2 given by (27), leading to the same symmetry constraint as in the previous case ($\tilde{c} = 1/c$). With the help of the same auxiliary variables (32)–(33), we arrive at the following OΔE

$$w_{(1,0)} = \frac{\alpha\,w + \beta}{\gamma\,w + \delta}\,, \tag{38}$$

where $w = x\,x_{(-1,0)}$ and the parameters are given by $\alpha = -\delta = r\,\tilde{c}$, $\beta = r^2(1+\tilde{c})$, $\gamma = 1 - \tilde{c}$ and $r = p - q$. Equation (38) is a discrete Riccati equation which can be solved explicitly, by using the symmetry generator

$$X = \big(\gamma w^2 + (\delta - \alpha)w - \beta\big)\partial_w\,. \tag{39}$$

It should be noted that, when $\tilde{c} = -1$, the invariant solutions obtained above correspond to the periodic reduction $f_{(-1,1)} = f_{(1,-1)}$.

4 Concluding Remarks

The main purpose of this work was to demonstrate that the notions of symmetry and invariance on the discrete level arise naturally from the interplay between PΔEs and PDEs that share a common set of solutions. Moreover, certain symmetry characteristics which admit the aforementioned cross configuration can be used to derive invariant solutions, in exact analogy with the invariant solutions of the continuous PDEs. In connection with the latter issue, recently in [11], a parameter family of discrete OΔEs which are compatible with the full Painlevé VI differential equations was derived. More recently in [19], the discrete multi-field Boussinesq system and the compatible PDEs were investigated. It was shown that scaling invariant solutions of the relevant PDEs are built from solutions of higher Painlevé equations, which potentially lead to solutions in terms of new transcendental functions. Thus, it is even more interesting to find the compatible discrete reduced system.

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