Active array imaging in free space

In the case of active imaging the array elements act as sources and receivers and the object that we wish to image is a scatterer. The geometry of the problem is depicted in Figure 1. We dispose of a linear array (in 2d) which sends the pulse $f(t, \vec{\mathbf{x}}_s)$ from the sources

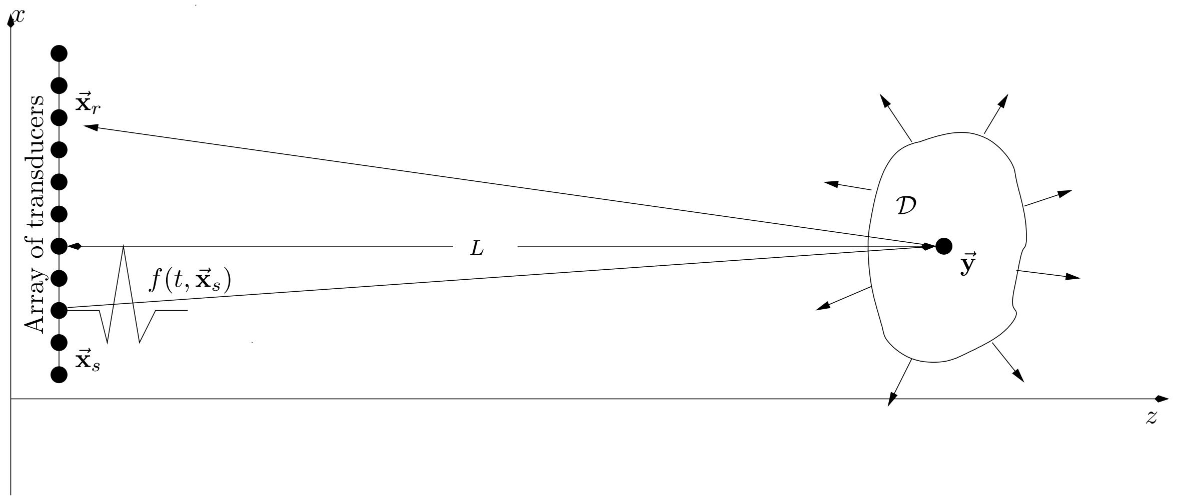

Figure 1: Setup for imaging a distributed scatterer $\mathcal{D}$ with a passive array of transducers in free space.

and records the data at the receivers. We assume here that the sources and receivers are point transducers located at the same points, denoted $\vec{\mathbf{x}}_s$, $s = 1, \ldots, N$ for the sources and $\vec{\mathbf{x}}_r$, $r = 1, \ldots, N$ for the receivers. The aperture of the array is $a = (N-1)\ h$, with h the array pitch, that is, the distance between the receiver elements. The data recorded at the array is the acoustic pressure field $p(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, t)$ recorder at receiver $\vec{\mathbf{x}}_r$ when the pulse $f(t, \vec{\mathbf{x}}_s)$ is emitted from the source located at $\vec{\mathbf{x}}_s$.

In imaging we are interested in solving the following problem:

Problem 1 *Find the support D of the scatterer given the array response matrix $p(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, t)$, for $s, r = 1, \ldots, N$ and t in some time interval $[0, T]$.*

We will assume that the source function $f(t, \vec{\mathbf{x}}_s)$ is of the following form,

$$f(\vec{\mathbf{x}}, t) = f(t)\delta(\vec{\mathbf{x}} - \vec{\mathbf{x}}_s). \tag{1}$$

To produce numerically the array data we can solve the wave equation either in the time or in the frequency domain. We can also use the following expression

$$\widehat{p}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega) = \widehat{f}(\omega) \int_D d\vec{\mathbf{y}} \widehat{G}_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}, \omega) \widehat{G}_0(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}, \omega), \tag{2}$$

with

$$\widehat{G}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega) = \frac{e^{i\ \omega \frac{|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}{c_0}}}{4\pi\,|\vec{\mathbf{x}}-\vec{\mathbf{y}}|},$$

being the Green's function in the homogeneous background medium. For this project take $c_0 = 1500m/s$.

To produce the images, you should use the Kirchhoff migration imaging functional,

$$\mathcal{I}^{\mathrm{KM}}(\vec{\mathbf{y}}^S) = \int d\omega \sum_{r=1}^{N}\sum_{s=1}^{N} \overline{G_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S, \omega)}\widehat{p}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)\overline{G_0(\vec{\mathbf{x}}_s, \vec{\mathbf{y}}^S, \omega)} \tag{3}$$

Source function For the imaging part assume that the array data are known at the frequency range $[f_0 - B/2, f_0 + B/2]$, with f_0 the central frequency (recall that $\omega = 2\pi f$). For the construction of the data you will need to program the following source function,

$$\widehat{f_2}(\omega) = e^{-(\omega-\omega_0)^2/(2\sigma^2)}$$

with $\sigma = \pi B/3$.

1. **Linear array** Consider a linear array with $N = 51$ elements and array pitch $h = 0.5$m. The location of the array elements is $\vec{\mathbf{x}}_r = (x_r, z)$ with $z = 10$m and $x_r = 10+(r-1)\ h$ (in m). We call the direction x the cross-range and z the range.

   (a) **Point target** Consider the case of a point target located at $\vec{\mathbf{y}}^* = (22.5, L+10)$m, $L = 200$m.

      i. Construct the KM image using only one frequency, $f_0 = 1.5$kHz. What do you observe? What is the resolution in range? in cross range? Compare with the theory. You can use either one source on the array or many sources. Do you get any benefit by using more sources?

      ii. Add white measurement noise to the data with different values of SNR. Take for example $SNR = 10, 0, and -10$ dB. Look at the paper [2] for details about the SNR (cf. also [3]). Compare between one and N sources. Do you get any benefit by using more sources?

      iii. Construct the KM image using $f_0 = 1.5$kHz and $B = 1$kHz. Use one and N sources for different values of SNR. What do you observe?

      iv. For $f_0 = 1.5$kHz, $B = 1$kHz and the three SNR levels given above compute the singular value decomposition of the response matrix frequency by frequency. For the central frequency plot the sigular values of the response matrix. How many are significant? Do you see the effect of the noise on the singular values?

Plot also the first singular value as a function of frequency for the three SNR levels. What do you observe?

Use the subspace projection method that we described in the course (see also appendix A). Take $J(\omega) = 1$, *i.e.*, keep the strongest reflection at each frequency and do the KM image. Are the images better than the ones obtained in iii? Why?

(b) **Extended object** Consider the case of an extended object. Take for example, as domain D a crack parallel to the array with center $\vec{\mathbf{y}}^* = (22.5, L + 10)$m, $L =$ 200m and size $b = 16$m. Compute the noisy response matrix using $f_0 = 1.5$kHz, $B = 1$kHz and for three SNR levels $SNR = 10, 0, and - 10$ dB.

i. Construct the KM image for the three SNR levels. What do you observe?

ii. Compute the singular value decomposition of the response matrix frequency by frequency. For the central frequency plot the sigular values of the response matrix. How many are significant? Do you see the effect of the noise on the singular values? Plot also the first (second, third, ...) singular value as a function of frequency for the three SNR levels. What do you observe?

iii. Use the subspace projection method that we described in the course (see also appendix A). Take $J(\omega) = 1$, *i.e.*, keep the strongest reflection at each frequency and do the KM image.

iv. Can you chose the $J(\omega)$ appropriately frequency by frequency so as to reconstruct the boundary of the object (*i.e.*, the two end points of the crack)?

2. **Circular array - extended object** Consider a circular array with $N = 101$ (equidistant) elements. The location of the array elements is on a circle centered at zero with radius $r = 20$m. You might need to increase the number of array elements if the results are not very good, make some tests to decide. Use your experience from the source problem. Consider the case of an extended object. Take for example, as domain D a square with center $\vec{\mathbf{y}}^* = (0, 0)$m and size $b = 4$m. Compute the noisy response matrix for $f_0 = 1.5$kHz, $B = 1$kHz and for three SNR levels $SNR = 10, 0, and - 10$ dB.

(a) Construct the KM image for the three SNR levels. What do you observe?

(b) Compute the singular value decomposition of the response matrix frequency by frequency. For the central frequency plot the sigular values of the response matrix. How many are significant? Do you see the effect of the noise on the singular values? Plot also the first (second, third, ...) singular value as a function of frequency for the three SNR levels. What do you observe?

(c) Use the subspace projection method that we described in the course (see also appendix A). Take $J(\omega) = 1$, *i.e.*, keep the strongest reflection at each frequency and do the KM image.

(d) Can you chose the $J(\omega)$ appropriately frequency by frequency so as to reconstruct the boundary of the object ?

A The singular value decomposition of the response matrix

The singular value decomposition of the $N \times N$ response matrix $\hat{\Pi}(\omega) = \left\{\hat{\Pi}(\vec{\mathbf{x}}_r, \vec{\mathbf{x}}_s, \omega)\right\}$ at any frequency ω in the bandwidth is given by

$$\hat{\Pi}(\omega) = \sum_{j=1}^{N} \sigma_j(\omega)\hat{\mathbf{u}}_j(\omega)\hat{\mathbf{v}}_j^{\star}(\omega), \tag{4}$$

so that

$$\hat{\Pi}(\omega)\hat{\mathbf{v}}_j(\omega) = \sigma_j(\omega)\hat{\mathbf{u}}_j(\omega), \quad j = 1, \ldots, N. \tag{5}$$

Here the star stands for complex conjugate and transpose. The singular values $\sigma_j(\omega) \geq 0$ are in decreasing order and $\hat{\mathbf{u}}_j(\omega)$, $\hat{\mathbf{v}}_j(\omega)$ are the orthonormal left and right singular vectors, respectively.

Because the complex matrix $\hat{\Pi}(\omega)$ is symmetric, although not hermitian, we can determine the left singular vectors as the complex conjugates of the right ones. However, this is true only when the correct phase has been assigned to these vectors. The computation of the SVD with any public software returns

$$\hat{\mathbf{u}}_j(\omega) = e^{i\varphi_j(\omega)}\overline{\hat{\mathbf{v}}_j(\omega)}, \quad j = 1, \ldots, N, \tag{6}$$

with an ambiguous phase that is difficult to unwrap in a consistent manner across the bandwidth. Nevertheless, the projection matrices

$$\mathcal{P}_j(\omega) = \hat{\mathbf{u}}_j(\omega)\hat{\mathbf{u}}_j^{\star}(\omega) \tag{7}$$

onto the space spanned by the jth left singular vector have no phase ambiguities, and this is what we use in the algorithm described below.

A.1 Data filtering

Let us assume that the number N of array elements is large enough so that the number $n^{\star}(\omega)$ of significant singular values of $\hat{\Pi}(\omega)$ is smaller than N.

The filtered version of the response matrix, $D[\widehat{\Pi}(\omega);\omega]$, is defined in the following way,

$$D[\widehat{\Pi}(\omega);\omega] \; = \sum_{j=1}^{N} d_j(\omega)\mathcal{P}_j(\omega)\hat{\Pi}(\omega) \tag{8}$$

with $d_j(\omega) \geq 0$ the filter weights that we take as binary,

$$d_j(\omega) = \begin{cases} 1 & \text{if } j \in J(\omega) \\ 0 & \text{otherwise }, \end{cases} \tag{9}$$

for some set $J(\omega) \subset \{1, \ldots, N\}$ that determines which singular vectors of $\widehat{\Pi}(\omega)$ we keep. The simplest choice for the filter weights is to take $J^I(\omega) = \{1, 2, \ldots, N\}$ so that D becomes the identity. Another choice is to take $J^{detect}(\omega) = \{1\}$, i.e., keep the strongest reflection at each frequency. This is a very good method for detection because it is robust to noise. However, it is not a good method for imaging. That is because the largest singular vector corresponds in general to direct or specular reflections from the bulk of the object and imaging with it will not provide any information about the geometrical details of the object, such as the edges or the corners.

The filter $J^{detect}(\omega)$ is related to the DORT method [4] which is designed to selectively image or focus energy on well-separated point-like targets. It relies on the fact that the array response matrix for m such targets has rank m, and that each singular vector corresponds to a different target. For point-like targets that are not well separated, an optimization approach introduced in [1] can determine weights $d_j(\omega)$ that image the targets one by one in a robust way. This optimization approach is coupled with the adaptive CINT functional and therefore can be used in cluttered media. However, it does not generalize in an obvious way to extended reflectors.

To focus on the edges of an extended object we follow the approach in [3] and define the filter weights so that the normalized singular values $\sigma_j(\omega)/\sigma_1(\omega)$ of $\widehat{\Pi}(\omega)$ are in some interval $[a, b] \subset (0, 1)$,

$$J^{SM}(\omega; [a, b]) = \left\{ j \,\middle|\, \frac{\sigma_j(\omega)}{\sigma_1(\omega)} \in [a, b] \right\}. \tag{10}$$

Selectively imaging the edges of extended objects in homogeneous media with the subspace migration method was extensively studied in [3]. It was shown with numerical simulations that this imaging method masks the strong specular reflections from the bulk of the object and allows to image its edges quite effectively. It is also robust to noise for arrays that have a large number of sensors. The analysis of the imaging method was carried out in the Fraunhofer regime using the theory of generalized prolate spheroidal wave functions.

In [2] this method was applied for selectively image the edges of a crack. This allowed us to obtain a better estimate of its size especially in cluttered media.

References

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