Passive and Active array imaging in a cavity

In the case of passive imaging the array is acting as a receiver and the object that we wish to image as a source. For a linear array in a cavity, the geometry of the problem is depicted in Figure 1. The linear array is composed by point transducers located at $\vec{\mathbf{x}}_r$, $r = 1, \ldots, N$.

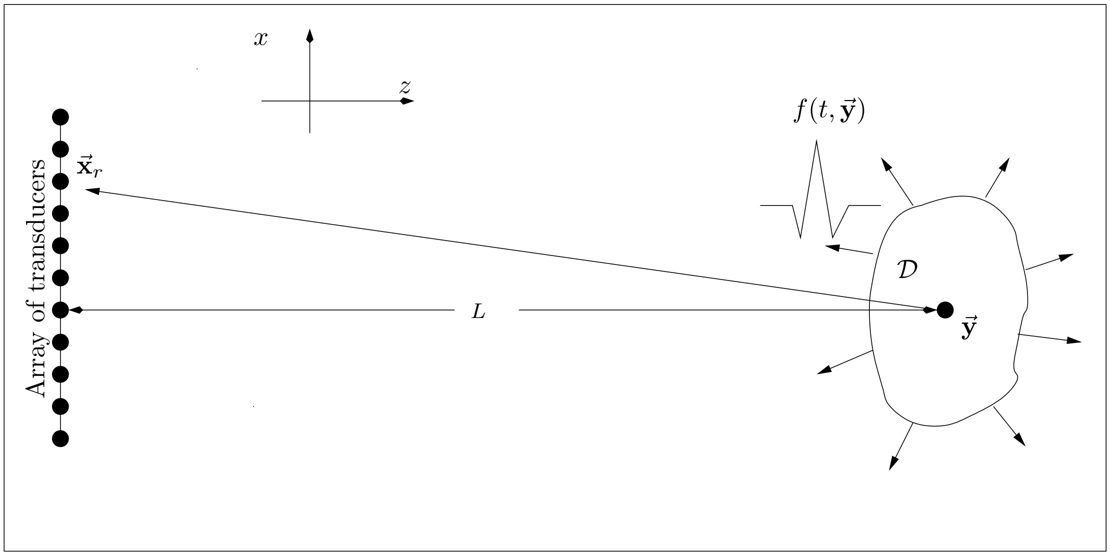

Figure 1: Setup for imaging a distributed source $\mathcal{D}$ with a passive array of transducers in a cavity.

The aperture of the array is $a = (N-1)\ h$, with h the array pitch, that is, the distance between two consecutive receiver elements. The data recorded at the array is the acoustic pressure field $p(\vec{\mathbf{x}}_r, t)$, with $p(\vec{\mathbf{x}}, t)$ the solution of the wave equation,

$$\frac{1}{c(\vec{\mathbf{x}})^2}\frac{\partial^2}{\partial t^2}p(\vec{\mathbf{x}}, t) - \Delta p(\vec{\mathbf{x}}, t) = f(\vec{\mathbf{x}}, t). \tag{1}$$

At the boundary of the cavity we consider homogeneous Dirichlet boundary conditions (*i.e.*, the pressure field is zero on the boundary of the domain). We assume that the source function $f(\vec{\mathbf{x}}, t)$ is of the following form,

$$f(\vec{\mathbf{x}}, t) = f(t)\,\mathbb{1}_D(\vec{\mathbf{x}}), \tag{2}$$

where $\mathbb{1}_D(\vec{\mathbf{x}})$ is the indicator function of the domain D.

Equivalently, in the frequency domain $\widehat{p}(\vec{\mathbf{x}}, \omega)$, the Fourier transform of $p(\vec{\mathbf{x}}, t)$, is the solution of the reduced wave (Helmholtz) equation

$$\Delta\widehat{p}(\vec{\mathbf{x}}, \omega) + \frac{\omega^2}{c(\vec{\mathbf{x}})^2}p(\vec{\mathbf{x}}, \omega) = -\widehat{f}(\omega)\,\mathbb{1}_D(\vec{\mathbf{x}}). \tag{3}$$

Remark 1 *In the wave equations (1) and (3), the wave propagation velocity depends, in general, on position. For this project we assume that it is constant* $c_0(\vec{\mathbf{x}}) = c_0 = 3000m/s$*.*

To produce numerically the array data we can solve the wave equation either in the time or in the frequency domain. We can also compute the solution using the following expression

$$\widehat{p}(\vec{\mathbf{x}}_r, \omega) = \widehat{f}(\omega) \int_D d\vec{\mathbf{y}} \widehat{G}_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}, \omega), \tag{4}$$

with $\widehat{G}_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}, \omega)$ the Green's function in the cavity. Note that the Green's function can be computed analytically for particular geometries, as for example in the case of a rectangular cavity.

To produce the images, you should use the Kirchhoff migration imaging functional,

$$\mathcal{I}^{\text{KM}}(\vec{\mathbf{y}}^S) = \int d\omega \sum_{r=1}^{N} \overline{\widehat{p}(\vec{\mathbf{x}}_r, \omega)} G_0(\vec{\mathbf{x}}_r, \vec{\mathbf{y}}^S, \omega) \tag{5}$$

Central frequency/Multiple frequencies For the imaging part assume that the array data are known at the frequency range $[f_0 - B/2, f_0 + B/2]$, with f_0 the central frequency (recall that $\omega = 2\pi f$). For the construction of the data you should consider the following source function,

$$\widehat{f}_1(\omega) = \mathbb{I}_{[\omega_0 - \pi B, \omega_0 + \pi B]}$$

or some (smooth) tapered version of $\widehat{f}_1(\omega)$ with support in the same bandwidth.

The length units in the following will be given in terms of the reference wavelength $\lambda_0 = f_0/c_0$.

Assume a rectangular cavity $[0, L_z] \times [0, L_x]$, with $L_z = 40\lambda_0$ and $L_x = 30\lambda_0$. If the domain is too big for computational reasons decrease everything analogously.

Questions

Consider a linear array with $N = 51$ elements and array pitch $h = \lambda_0/2$, the array aperture is $a = (N-1)h/2$. The location of the array elements is $\vec{\mathbf{x}}_r = (x_r, z)$ with $z = 2\lambda_0$ and $x_r = 15\lambda_0 + (r-1)\ h - a/2$.

1. **Point source** Consider the case of a point source located at $\vec{\mathbf{y}}^* = (20\lambda_0, 25\lambda_0)$.

   (a) Construct the KM image using only one frequency, $f_0 = 1.5$kHz. What do you observe? What is the resolution in range? in cross range? To see better the resolution in range you can plot the image as a function of range at the correct cross-range of the source. The same can be done with the cross-range. Increase the frequency to $f_0 = 3$kHz. What do you observe? Carry out a theoretical analysis of the resolution of the method.

(b) Add white noise to the data according to the model described by equations (1.2)-(1.3) of chapter 3 in the notes. Chose $\sigma_{\mathcal{N}}$ so that the SNR of the data takes the values $10, 0, -10$ dB. How does the SNR of the image depends on the SNR of the data? the number of receivers?

(c) Use multiple frequencies, what is the effect of the bandwidth on the resolution of the image? On the SNR?

(d) Change the array size, what do you observe? Can you find the location of the source with one array element? with two?

2. **Multiple point sources** Consider the case of two point sources located at distance $d = 2\lambda_0$. Consider three configurations: either the sources have the same range, or they have the same cross-range or they have different range and cross-range. Decrease the distance d, how far should the two sources be in order to be visible as two sources in the image? Can you find a configuration for which one source become invisible?

3. **Extended source** Consider the case of an extended source. Take for example, as domain D a disk with center $\vec{\mathbf{y}}^*$ and radius $b = 2\lambda_0$.

   (a) Construct the KM image using only one frequency, $f_0 = 1.5$kHz. What do you observe?

   (b) Construct the KM image using $f_0 = 1.5$kHz and $B = 1$kHz. What do you observe?

   (c) What happens for a different shape of D, a square or a rectangle?

4. **Consider an active configuration where the sources become point targets.**

   (a) Is there a difference between the active and passive case in terms of resolution?

   (b) To construct the image use either the full array data or just one column of the response matrix. Do you observe a difference?(in terms of resolution, SNR?)

   (c) Can you cook up a geometric configuration where one point target becomes invisible? Do to this use either the full array data or just one column of the response matrix to construct the image.

5. **Kirchhoff-Helmholtz identity** Use white noise sources on the array and record the solution of the wave equation $p(\vec{\mathbf{y}}_1, t)$ and $p(\vec{\mathbf{y}}_2, t)$ at two points $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. Compute the empirical cross-correlation between these two recordings

$$C_T(\tau, \vec{\mathbf{y}}_1, \vec{\mathbf{y}}_2) = \frac{1}{T} \int_0^T p(\vec{\mathbf{y}}_1, t) p(\vec{\mathbf{y}}_2, t + \tau) dt$$

   and compare it with the symmetrized Green's function between these two points. Consider different configurations of $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{y}}_2$. What do you observe?

All images should be computed on a square of size $20\lambda_0 \times 20\lambda_0$ with a discretization of $\lambda_0/4$ and centered at $\vec{\mathbf{y}}^*$.
The documents eigen.pdf and EFguide.pdf should be useful for this project.