Diffraction d’une onde acoustique par un objet

1 Physical problem and equations

We are interested in the problem of wave scattered by an object in a 2D homogeneous acoustic medium.

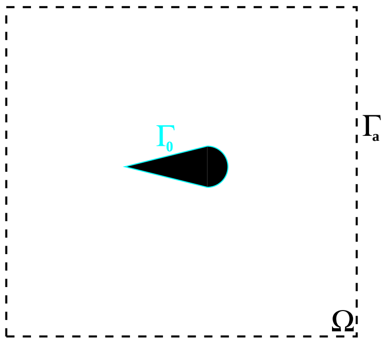

Figure 1: The geometry of the problem.

We assume that the pressure field u, satisfies the acoustic wave equation in the domain C (the domain exterior of the object) :

$$(1) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle u = f & \text{in} \quad C \times \mathbb{R} \\ u = 0 & \text{on} \quad \Gamma_0 \end{cases}$$

with the initial conditions :

$$(2) \qquad \begin{cases} u(x,0) \quad = \quad u_0(x) \quad \text{in } C \\ \dfrac{\partial u}{\partial t}(x,0) \quad = \quad u_1(x) \quad in\ C \end{cases}$$

and with boundary conditions on the exterior boundary Γ_a that we will precise later. In (1), c is the wave speed, which is here assumed constant, we can chose for example $c = 1$.

Energy

We define the energy of the system as the sum of kinetic and potential energy :

$$(3) \qquad E(t) = \int_\Omega \left(\frac{1}{2} \mid \frac{\partial u}{\partial t} \mid^2 + \frac{1}{2} \mid \nabla u \mid^2 \right) dx$$

Question 1.1 *Show that, if we consider the space($\Omega = R^2$, and there is no object) and if there is no external force ($f =$), the energy is conserved :* $\dfrac{dE(t)}{dt} = 0$

Boundary conditions

We consider now following absorbing boundary conditions on Γ_a :

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_a \tag{4}$$

where n is the exterior normal.

Question 1.2 *Compute the energy again. Show that in this case the energy is decreasing with time.*

We introduce the spaces $H = L^2(C)$ and

$$V = \{v \in H^1(C); v = 0 \ \ on \ \ \Gamma_0\}$$

Question 1.3 *Show that the variational formulation of problem (1), (2) and (4) is :*
Find $u(x,t) \in V$ such that :

$$\frac{d^2}{dt^2}\int_C uv dx + \frac{d}{dt}\int_{\Gamma_a} uv d\gamma + \int_C \nabla u.\nabla v dx = \int_C f v dx \tag{5}$$

for all $v \in V$.
We admit that we seek the solution of this problem in the space $L^2(0,T;V) \cap C^0(0,T;H)$.

2 Plane wave analysis

We are interested here in the problem of wave propagation in ($\mathbb{R}^2$) :

$$\begin{cases} \text{Find } u \text{ in } H^1(\mathbb{R}^2) \text{ such that:} \\ \dfrac{d^2u}{dt^2} - c^2 \triangle u = 0 \end{cases} \tag{6}$$

We define the plane wave solutions of (6) in the form :

$$u = exp(i(\omega t - \vec{k} \cdot \vec{x})) \tag{7}$$

where

- $\vec{k}$ is the wave vector (that indicates the direction of propagation).
- ω is the frequency

Question 2.1 *Show that searching plane wave solutions of the form (7) implies that we seek for ω that satisfies :*

$$\omega^2 = c^2(k_1^2 + k_2^2) \tag{8}$$

Write corresponding dispersion relation (that relates the $|\vec{k}|$ *to* ω*. Compute the phase velocity defined by :*

$$V = \frac{\omega}{|\vec{k}|}$$

Remark that the phase velocity is independent of frequency. We say that the wave equation is not dispersive.

3 Approximation

3.1 Semi-discretization in space

In the following, we assume that the union of C and the object is the domain $\Omega = [0, a] \times [0, b]$. We consider a mesh of C that takes into account the shape of the object, $\Omega = \cup_{j=1}^{Nel} T_j)$, with Nel the number of triangles.

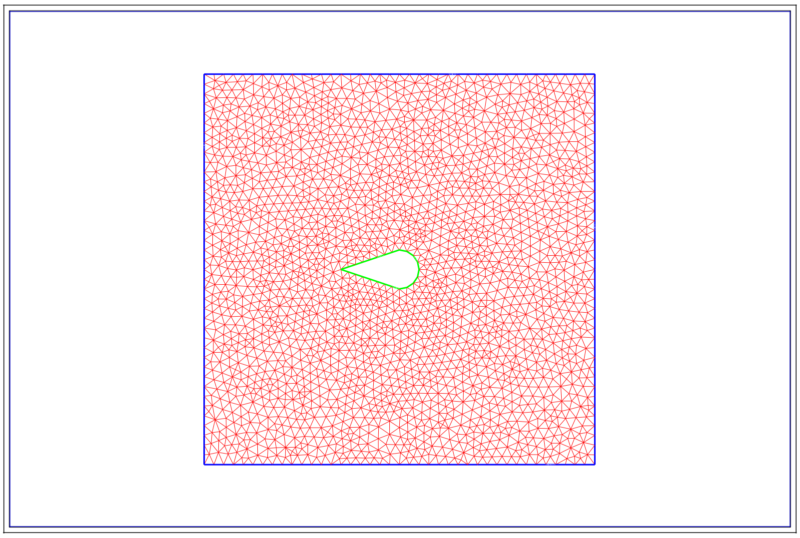

Figure 2: An example of mesh

We denote $M_{i=1..N_t}$ the nodes of the mesh (summits of the elements). We note H_h^1 the finite dimensional subspace of $H^1(C)$, of P^1 finite elements, , i.e.,

$$H_h^1 = \left\{ v \in C^0(\bar{C}), v_{/T_j} \in P^1(T_j) \forall j \right\}$$

(w_j) a basis of H_h^1, defined by :

$$w_j(M_i) = \delta_{ij} \;\; 1 \leq i, j \leq N_t$$

and V_h the sub-space of V of dimension $N_h = N_t - N_0 \leq N_t$, where N_0 are the nodes on the boundary of the object :

$$V_h = \left\{ v \in H_h^1; v = 0 \;\; on \;\; \Gamma_0 \right\}$$

Question 3.1 *Write the discrete variational formulation by decomposing the solution u_h on the basis of V_h :*

$$u_h(x,t) = \sum_{j=1}^{N_h} u_j(t) w_j(x)$$

Show that the problem can be written in the following matrix form :

$$M_h \frac{d^2U}{dt^2} + B_h^\Gamma \frac{dU}{dt} + K_h U = F_h \tag{9}$$

where the matrices M_h (mass), K_h (stiffness) and B_h^Γ (matrix associated with the absorbing boundary conditions) will be precised and U contains all the degrees of freedom :

$$U = (u_1 ... u_{N_h})$$

Question 3.2 *Write an algorithm that permits the computation of the elementary mass and stiffness matrices. The stiffness matrix should be computed exactly. For the mass matrix, we propose the use of the following quadrature formula :*

$$\int_T f dxdy \approx \frac{\mid T \mid}{4} (f_1 + f_2 + f_3) \Delta x \Delta y, \tag{10}$$

where f_i is the value of the function on the node i (the summits of the elements), $\mid T \mid$ is the area of T. Remark that the resulting mass matrix M_h^{ap} is diagonal.

Question 3.3 *Write an algorithm for assembling the mass and stiffness matrices.*

Definition of the matrix B_h^Γ.

This is a mass matrix on the boundary of the domain. The computation of this matrix uses only the nodes on the boundary Γ_a. Remark that the restriction of basis functions ϕ_i on the boundary is a P^1 of one variable.

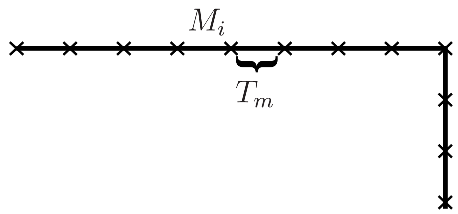

Figure 3: Example of mesh on the boundary Γ_a

The terms

$$B_{kl}^m = \int_{\Gamma_a} \phi_l^m \phi_k^m d\gamma$$

will be computed using the following quadrature formula :

$$\int_I f = \frac{\mid I \mid}{2} (f_1 + f_2) \tag{11}$$

where $f_{i=1,2}$ is the value of f on the points $i = 1, 2$ and $\mid I \mid$ is the length of the interval I.

Question 3.4 *Write an algorithm which permits the computation of the matrix B_h^Γ. Remark that the matrix B_{ap}^Γ computed using (11) is diagonal.*

3.2 Discretization in time

Question 3.5 *Using a central finite differences scheme for the time discretization we obtain,*

$$M_h^{ap}\frac{U^{n+1}-2U_n+U^{n-1}}{\Delta t^2}+B_{ap}^{\Gamma}\frac{U^{n+1}-U^{n-1}}{2\Delta t}+K_hU^n=F_h^n \tag{12}$$

where Δt is the discretization time and U^k is the solution at time $k\Delta t$.

3.3 Numerical dispersion analysis

To study the dispersion of the numerical scheme using P^1 finite elements we are interested in plane waves that propagate in the whole space for $f = 0$. We consider a regular mesh composed by triangles of size h :

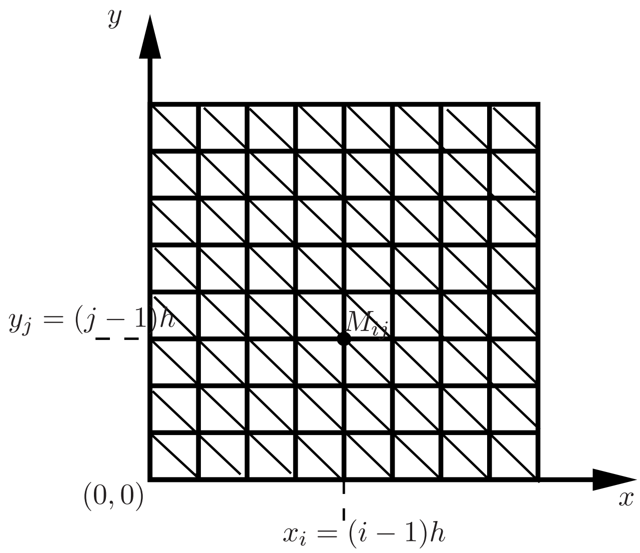

Figure 4: Example of a regular mesh

Consider the summit M_{ij} of the mesh.

Question 3.6 *Write the equations obtained on the pressure field (u) on this point (we enumerate from 0 to 6 the nodes M_{ij}, see figure 5)).*

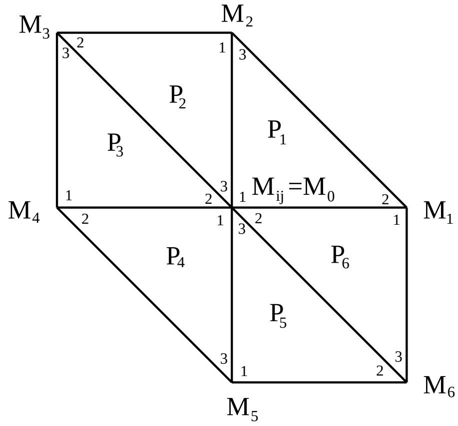

Figure 5: Numeration of nodes

Show that the equations can be written in the form :

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} + K_h u_h^n = 0 \tag{13}$$

where $B_h u_h^n$ *is*

$$\left\{ \quad K_h u_h^n \quad = \quad \frac{1}{h^2} \left\{ 4u_{i,j}^n - \left(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n \right) \right\} \right. \tag{14}$$

We obtained in this way an equivalent finite difference scheme. To obtain the numerical dispersion relation we seek solutions of the form :

$$u_h = exp[i(\omega n \Delta t - ik_1 h - jk_2 h)] \tag{15}$$

Question 3.7 *Show that the numerical dispersion relation is :*

$$\frac{1}{\Delta t^2} \sin^2\left(\frac{\omega \Delta t}{2}\right) - \frac{1}{h^2}\left(\sin^2\left(\frac{k_1 h}{2}\right) + \sin^2\left(\frac{k_2 h}{2}\right)\right) = 0 \tag{16}$$

We can show that a necessary condition for the scheme to be stable is that ω is real.

Question 3.8 *Write the condition* $\omega \in R$ *(ie,* $\sin^2(\omega \Delta t/2) \in [0,1]$*) in the form* $f(\beta_1, \beta_2) \in [0,1]^2, \quad \forall k_1, k_2$ *with* $\beta_1 = \sin^2(k_1 h/2)$ *and* $\beta_2 = \sin^2(k_2 h/2)$*. Studying the variations of this function* f *on the square* $[0,1]^2$*, deduce that this condition implies the following CFL condition :*

$$\frac{\Delta t}{h} \leq \frac{\sqrt{2}}{2} \tag{17}$$

4 Numerical study of dispersion

We study here the dispersion of the numerical scheme. We show that for the continuous problem we have $V = \dfrac{\omega}{|\vec{k}|} = c = 1$. The phase velocity for the discrete scheme is determined by

$$V^{num} = \frac{\omega^{num}}{|\vec{k}|}$$

where $\omega^{num}(\mid \vec{k} \mid)$ is the solution of the numerical dispersion relation :

$$\frac{2}{\Delta t^2}(1 - cos\omega^{num}\Delta t) = \hat{K}_h(k) = s_h(\vec{k}) \tag{18}$$

Lets define

$$q(\vec{k}) = \frac{\omega^{num}(\vec{k})}{\mid \vec{k} \mid V}$$

and

- the wavelength : $l = \dfrac{2\pi}{\omega}$.
- the number of points per wavelength : $G = \dfrac{l}{h} = \dfrac{2\pi}{\mid \vec{k} \mid h}$
- the inverse of G : $H = 1/G$.

Question 4.1 *For a given direction of propagation $\vec{k} = \mid \vec{k} \mid (cos\theta, sin\theta)$, we replace in s_h :*

$$k_1 = 2\pi\frac{H}{h}cos(\theta),\ k_2 = 2\pi\frac{H}{h}sin(\theta),\quad we\ have\ \mid \vec{k} \mid = 2\pi\frac{H}{h}$$

We can take $h = 1$. Write a program that computes the numerical phase velocity. Plot q_h as a function of H (for $H \in [0., 0.5]$) for different values of θ $\theta = 0, \frac{\pi}{12}, \frac{\pi}{6}$ and $\frac{\pi}{4}$.

5 Simulation

Choice of the la source
To simulate a point source located at (x_s, y_s), we can take,

$$\left\{\begin{array}{l} F(x,y,t) = f(r)g(t) \\ \text{ where } f(r) \text{ is a radial function : } f(r) = (1 - \dfrac{r^2}{a^2})^3 1_{B_a} \\ r = \sqrt{(x - x_s)^2 + (y - y_s)^2},\ a = 5\Delta x_{min} \\ \text{ where } \Delta x_{min} \text{ is the smallest distance between two nodes in the mesh} \\ 1_{B_a} \text{ is the indicator function of the ball centered at } (x_s, y_s) \text{ with radius } a \\ g(t) = \left\{\begin{array}{ll} -2(\pi f_0)^2(t - t_0)\exp{(-\pi f_0(t - t_0))^2} & t \in [0, t_1] \\ 0 & elsewhere \end{array}\right. \\ \text{with } f_0 = \dfrac{1}{G} \text{ the frequency of the source} \\ t_0 = \dfrac{1}{f_0},\ t_1 = \dfrac{2}{f_0} \end{array}\right. \tag{19}$$

and zero initial conditions,

$$\left\{\begin{array}{l} u_0(x,y) = 0 \\ u_1(x,y) = 0 \end{array}\right. \tag{20}$$

To compute F_h we decompose $f(r)$ on the basis of P^1 finite elements,

$$f_h(x,t) = \sum_{j=1}^{N_h} f_j w_j(x)$$

where f_j is the value of the function $f(r)$ on the point M_i and we use the quadrature formula (10) to compute F_h. Remark that we obtain:

$$F_h = M_h\,[f]$$

with M_h the diagonal mass matrix and $[f] = (f_1, f_2, .., f_{N_h})$ the vector that contains the values of the function $f(r)$ on the nodes of the mesh $M_{i=1,..,N_h}$. Therefore, we have,

$$F_h^n = F_h g(n\Delta t),\ \ \Delta t = \alpha \Delta x_{min}$$

Question 5.1 *Write a numerical code that permits to compute the wave equation with Dirichlet and absorbing boundary conditions. Use your code to do the following simulations :*

Question 5.2 Simplified case

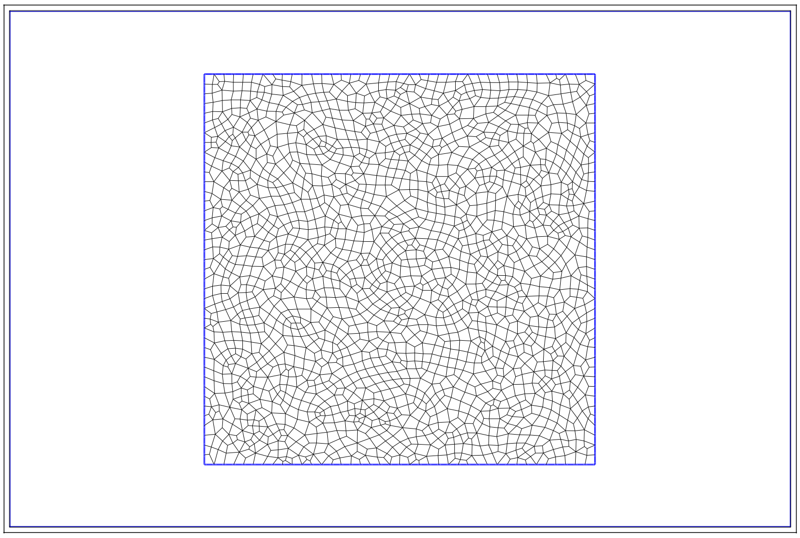

Figure 6: The case without object

a/ *Solve the problem without object and with homogeneous Dirichlet conditions on the boundary of the domain.*

b/ *Show the solution at different times on the whole domain. Also plot the solution as a function of time at a few points in the domain.*

c/ *Observe the influence of the value of G and $\alpha \frac{\Delta t}{\Delta x}$ on the solution. Do you observe instabilities for some values of α, is this the same as the value suggested by the theory?*

d/ *Observe the numerical anisotropy: what is the shape of the wavefront? When h goes to 0, are the wavefront circles?*

Absorbing boundary conditions

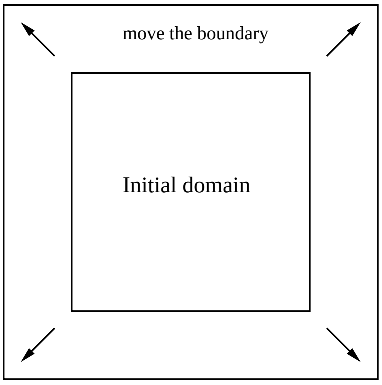

Figure 7: Domain with absorbing boundary

Question 5.3 *Solve the problem with ABC. Move the boundary further away. What do you observe?*

Smooth object

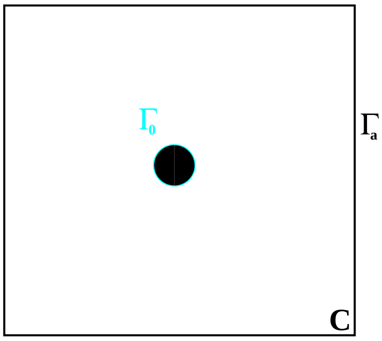

Figure 8: An object with smooth boundary

Question 5.4 *Solve the scattering problem for an object with smooth boundary. Show the solution at different times on the whole domain. Also plot the solution as a function of time at a few points in the domain.*

Non-smooth object

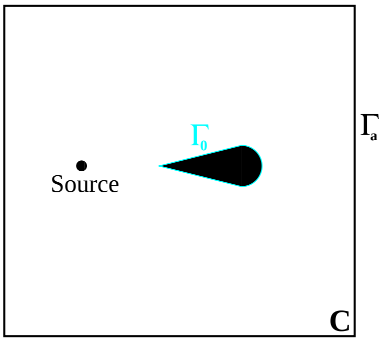

Figure 9: Object with non-smooth boundary

Question 5.5 *Solve the scattering problem for an object with non-smooth boundary. Show the solution at different times on the whole domain. Also plot the solution as a function of time at a few points in the domain.*