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On a Family of Cubics

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We consider the family of cubic Thue equations $x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$, and we give all its solutions for $n \geqslant 3.67 \cdot 10^{32}$.

1. INTRODUCTION

In recent years, considerable progress towards the practical solution of Thue equations has been made (here by "practical solution" we mean "the finding of all solutions explicitly").

This is based on the theory of linear forms in logarithms (see [Ba, Wa, Bl1, MW]) in combination with some computational techniques. Examples of solutions in this spirit are given by W. J. Ellison *et al.* [El], Steiner [St], Blass *et al.* [Bl2], Pethö and Schulenberg [PS], and Tzanakis and de Weger [TW]. In all these examples specific equations are solved and for every equation separately many calculations must be done, depending heavily on each particular equation. It seems therefore very difficult to find all the solutions of a parametrized family of Thue equations. Recently, however, E. Thomas [Th1] discovered parametrized families of cubic Thue equations, $\Phi_n(x, y) = 1$, depending on a positive integral parameter n, all of whose solutions are explicitly known provided that n is larger than some explicit constant. In his paper, E. Thomas proves that if $4 \leqslant n \leqslant 10^3$ or $n \geqslant 1.4 \cdot 10^7$, then the only solutions of the equation

$$x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 = 1$$

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are $(x, y) = (0, 1)$, $(1, 0)$, $(-1, 1)$. Very recently, Mignotte [Mi] filled the gap left in the above paper, while the cases $n = 0, 1, 2$ have been studied earlier. To the best of our knowledge, this is the first example of a parametrized family of Thue equations ever solved.

In Thomas' example, some special properties of that family of cubics are very useful, in particular, the fact that the corresponding cubic fields are Galois fields. In a second paper [Th2], Thomas considered other families of cubic equations $\Phi_n(x, y) = 1$, without such "good properties" as in [Th1] and was again able to prove that these equations have only trivial solutions (i.e., solutions with $xy = 0$) if n is larger than some effectively computable integer. Such forms he called stably trivial. It is worth noting that, in all examples studied by Thomas, the associated cubic fields are Galois.

In the present paper, we consider a non-Galois family of cubic Thue equations $\Phi_n(x, y) = 1$. The associated fields $\mathbf{Q}(\theta_n)$, where $\Phi_n(\theta_n, 1) = 0$, are totally real. One certainly crucial fact is that there is one pair of (rational) formulas $(\eta(\theta_n), \varepsilon(\theta_n))$ giving a pair of fundamental units of $\mathbf{Q}(\theta_n)$, for every n. Then our Thue equation is transformed into

$$x - y\theta_n = \pm\eta(\theta_n)^a\, \varepsilon(\theta_n)^b, \tag{1}$$

where (a, b) is an unknown pair of rational integers. Put $A = \max\{|a|, |b|\}$. By the algorithm of Tzanakis and de Weger [TW], we get an upper bound $U(n)$ of the number A in terms of n. Up to this point the only extra fact is the uniform shape of the units for every value of n. The next stage is a very crucial one: we prove that if $A \geqslant A_0$, where A_0 is very small, then $A \geqslant L(n)$, where the function L is explicitly known. At present, it is not clear under what circumstances we have such a situation. Anyway, in the specific example we consider we get such a function $L(n)$ if $A \geqslant 3$ and moreover (this is the important point) we have $L(n) > U(n)$ for large n. Therefore, if $n \geqslant N$, where N can be explicitly computed, the only solutions of the Thue equation $\Phi_n(x, y) = 1$ are those given by (1), when $A < A_0$ (in our case $A \leqslant 2$). This leads to the complete solution of $\Phi_n(x, y) = 1$ for $n \geqslant N$.

We have chosen the forms $\Phi_n(x, y) = x^3 - nx^2y - (n+1)xy^2 - y^3$. Note that each equation $\Phi_n(x, y) = 1$ has the five solutions, $(x, y) = (1, 0)$, $(0, -1)$, $(1, -1)$, $(-n-1, -1)$, $(1, -n)$.

According to a conjecture of Pethö [P] based on extensive computations, for any irreducible cubic form $\Phi_n(x, y) \in Z[x, y]$ with positive discriminant $\neq 49, 81, 148, 257, 361$, the equation $\Phi_n(x, y) = 1$ has at most five solutions. In our theorem in Section 5 we prove that, indeed, the five solutions mentioned above are the only solutions of the equation

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1, \tag{$*$}$$

if $n \geqslant 3.67 \cdot 10^{32}$, in accordance to Pethö's conjecture.

2. Preliminaries to the Solution of $x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$

We work in the field $K = \mathbf{Q}(\xi)$, where $\xi^3 - n\xi^2 - (n+1)\xi - 1 = 0$ (clearly ξ and K depend on n). The title equation is equivalent to "$x - y\xi$ is a unit of K."

The discriminant of ξ is $n^4 + 2n^3 - 5n^2 - 6n - 23 = (n^2 + n - 3)^2 - 32$, and hence it is positive for $n \geqslant 4$. For such n we known two fundamental units in K: put $\xi = \lambda^{-1} - 1$. Then $K = \mathbf{Q}(\lambda)$ and $\lambda^3 - (n+2)\lambda^2 + (n+3)\lambda - 1 = 0$, and therefore, by Thomas' paper [Th3], a pair of fundamental units is $\lambda, \lambda - 1$, i.e., $1/(1+\xi)$ and $(-\xi)/(1+\xi)$. From this it follows that $\xi, \xi + 1$ is a pair of fundamental units of K. Then, by (1), $x - y\xi = \pm\xi^a(1+\xi)^b$ for some $a, b \in \mathbf{Z}$. Since the norms of ξ and $1 + \xi$ are $+1$, the minus sign is excluded and

$$x - y\xi = \xi^a(1+\xi)^b. \tag{2}$$

Finally we put $A = \max\{|a|, |b|\}$.

3. An Upper Bound in Terms of n

We number the conjugates ξ_i of ξ in such a way that the following relations hold:

$$n + 1 < \xi_1 < n + 2, \qquad -\frac{n}{n+1} < \xi_2 < -\frac{n-1}{n}, \qquad -\frac{1}{n-1} < \xi_3 < -\frac{1}{n}. \tag{3}$$

A first step towards the solution of (2) is to apply the algorithm of Tzanakis and de Weger [TW] in order to compute an upper bound C_9 of A (here, we follow the notations of [TW]). This is accomplished by straightforward computation (using the estimates (3)) of a series of constants given by explicit formulas in that paper. In our case, under the assumption that $n > 10^8$, the series of constants is

$$\begin{aligned}
&C_1 = 4.000001/n, \qquad Y_0 = 1, \qquad Y_1 = 1, \qquad C_2 = 0.499999,\\
&C_3 = 1.00000009n, \qquad C_4 = 1.00000017n, \qquad Y_2^* = 3, \qquad Y_2' = 3,\\
&C_5 = 2.00000001/\log n, \qquad C_6 = 11.12004n^3,\\
&C_7 = 1.0717 \cdot 10^{27} \cdot \log^3 n \cdot \log\log n, \qquad C_8 = 2.2490587 \cdot \log\log n.\\
&A < C_9 = 3.6922806 \cdot 10^{28} \cdot \log^2 n \cdot (\log\log n)^2.
\end{aligned} \tag{4}$$

4. A Lower Bound of Λ in Terms of n

Referring again to Tzanakis and de Weger [TW], we have

$$0 < |\Lambda| < C_6 \cdot \exp\left(-\frac{3A}{C_5}\right) < \frac{11.12004}{n^{1.499999992A-3}}, \tag{5}$$

where

$$\Lambda = \log\left|\frac{\xi_{i_0}-\xi_j}{\xi_{i_0}-\xi_k}\right| + a\log\left|\frac{\xi_k}{\xi_j}\right| + b\log\left|\frac{1+\xi_k}{1+\xi_j}\right|$$

and i_0, j, k is a permutation of $\{1, 2, 3\}$, such that, once i_0 is chosen, the values of j and k are arbitrary.

For simplicity, we put

$$\delta = \left|\frac{\xi_{i_0}-\xi_j}{\xi_{i_0}-\xi_k}\right| \qquad \text{and} \qquad \varepsilon = 1+\xi,$$

so that

$$\Lambda = \log\delta + a\log\left|\frac{\xi_k}{\xi_j}\right| + b\log\left|\frac{\varepsilon_k}{\varepsilon_j}\right|.$$

We distinguish three cases (assuming $n > 10^8$), where i_0 is chosen according to Lemma 1.1 in [TW].

Case 1. $i_0 = 1$.

In this case, let us choose $j = 3$ and $k = 2$. We have $\log\delta = \log(1-z)$, where

$$z = \left|\frac{\xi_3-\xi_2}{\xi_1-\xi_2}\right|.$$

Using (3), we see that $0.99999997/(n+3) < z < 1/n$, and, in view of the elementary inequality $z < -\log(1-z) < 2z$, we obtain

$$-\frac{2}{n} < \log\delta < -\frac{0.99999994}{n}. \tag{6}$$

Also, in view of (5), $(n-1)^2/n < |\xi_2/\xi_3| < n^2/(n+1)$, which implies

$$\log n - \frac{2.00000003}{n} < \log\left|\frac{\xi_2}{\xi_3}\right| < \log n - \frac{0.99999999}{n}. \tag{7}$$

Finally, $\log |\varepsilon_k/\varepsilon_j| = -\log |(1+\xi_3)/(1+\xi_2)|$, and, in view of (5), we have

$$\frac{n^2-2n}{n-1} < \left|\frac{1+\xi_3}{1+\xi_2}\right| < \frac{n^2-1}{n},$$

and

$$-\log n + \frac{1}{n^2} < \log \left|\frac{\varepsilon_2}{\varepsilon_3}\right| < -\log n + \frac{1.00000005}{n}. \tag{8}$$

From (6), (7), (8) we get

$$\log \delta = \frac{z_0}{n}, \qquad -2 < z_0 < -0.99999994,$$

$$\log \left|\frac{\xi_2}{\xi_3}\right| = \log n - \frac{z_1}{n}, \qquad 0.99999999 < z_1 < 2.00000003,$$

$$\log \left|\frac{\varepsilon_2}{\varepsilon_3}\right| = -\log n + \frac{z_2}{n}, \qquad 0 < z_2 < 1.00000005.$$

Therefore,

$$\Lambda = (a-b)\log n + \frac{z_0 - az_1 + bz_2}{n}.$$

Suppose that $\Lambda \leqslant (n \log n)/4$. If $a \neq b$, then

$$|\Lambda| \geqslant |a-b| \cdot \log n - \frac{|z_0 - az_1 + bz_2|}{n} \geqslant \log n - \frac{2}{n} - \frac{3.00000008}{n} A$$

$$\geqslant 0.24999997 \log n - \frac{2}{n} > 0.24999995 \log n,$$

which contradicts (5) if $A \geqslant 3$. We conclude therefore that $3 \leqslant A \leqslant (n \log n)/4$ implies $a = b$. In this case

$$\Lambda = \log \delta + a \log \left|\frac{\xi_2 \varepsilon_2}{\xi_3 \varepsilon_3}\right| = \log \delta + a\left(\log \frac{1}{|\xi_3 \varepsilon_3|} - \log \frac{1}{|\xi_2 \varepsilon_2|}\right).$$

Now the relations (3) do not provide us with a satisfactory estimate of the coefficient of a, because they give a *positive* upper bound and a *negative* lower bound, and thus they do not exclude the case of an "extremely small" coefficient. As we need a higher accuracy, we calculate the defining equation of $|\xi\varepsilon|^{-1}$ and from it the first terms of the continued fraction for $|\xi_2\varepsilon_2|^{-1}$ and $|\xi_3\varepsilon_3|^{-1}$. We find

$$|\xi_2 \varepsilon_2|^{-1} = [n+1, 1, n-1, 1, \ldots],$$

from which

$$\frac{n^2+3n+1}{n+1}-\frac{1}{(n+1)^2}<\frac{1}{|\xi_2\varepsilon_2|}<\frac{n^2+3n+1}{n+1},$$

and

$$|\xi_3\varepsilon_3|^{-1}=[n+1, n-1, 1, ...],$$

from which

$$\frac{n^2+n+1}{n}<\frac{1}{|\xi_3\varepsilon_3|}<\frac{n^2+n+1}{n}+\frac{1}{n^2}.$$

It follows that

$$-\log n-\frac{2}{n}+\frac{1}{n(n+1)}< -\log\frac{1}{|\xi_2\varepsilon_2|}< -\log n-\frac{2}{n+1}+\frac{1}{(n+1)^2},$$

$$\log n+\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{n^3}-\frac{1}{2n^4}<\log\frac{1}{|\xi_3\varepsilon_3|}<\log n+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3},$$

and hence

$$-\frac{1}{n}< -\log\frac{1}{|\xi_2\varepsilon_2|}+\log\frac{1}{|\xi_3\varepsilon_3|}< -\frac{0.99999996}{n}.$$

Since $A\geqslant 3$ and $|\log\delta|<2/n$, we have $|\Lambda|>3\cdot 0.99999996/n-2/n$, which contradicts (5).

We conclude, therefore, that, in Case 1, there are no solutions of (2) in the range $3\leqslant A\leqslant (n\log n)/4$.

Case 2. $i_0=2$.

We choose now $j=1$ and $k=3$. Using (3) we get

$$\begin{aligned}
&\log\delta=\log n+\frac{z_0}{n}, && 2.99999997<z_0<5.00000006,\\
&\log\left|\frac{\xi_3}{\xi_1}\right|=-2\log n+\frac{z_1}{n}, && -2<z_1<0.00000002, \qquad (9)\\
&\log\left|\frac{\varepsilon_3}{\varepsilon_1}\right|=-\log n-\frac{z_2}{n}, && 2<z_2<4.00000003.
\end{aligned}$$

Therefore,

$$\Lambda=(1-2a-b)\log n+\frac{z_0+az_1-bz_2}{n}.$$

Suppose that $\Lambda \leqslant (n \log n)/7$. If $1-2a-b \neq 0$, then

$$|\Lambda| \geqslant \log n - \frac{5.00000006}{n} - \frac{6.00000003}{n} \Lambda \geqslant 0.1427 \log n,$$

which contradicts (5) if $\Lambda \geqslant 3$. Therefore, in the range $3 \leqslant \Lambda \leqslant (n \log n)/7$ we must have $b = 1-2a$. In this case, $a \neq 0$ and

$$\begin{aligned} \Lambda &= \log \delta + a \log \left|\frac{\xi_3}{\xi_1}\right| + (1-2a) \log \left|\frac{\varepsilon_3}{\varepsilon_1}\right| \\ &= \left(\log \delta + \log \left|\frac{\varepsilon_3}{\varepsilon_1}\right|\right) + a\left(\log \left|\frac{\xi_3}{\xi_1}\right| - 2 \log \left|\frac{\varepsilon_3}{\varepsilon_1}\right|\right). \end{aligned}$$

In view of (9), the coefficient of a is greater than $2/n$, and

$$-\frac{1.00000006}{n} < \log \delta + \log \left|\frac{\varepsilon_3}{\varepsilon_1}\right| < \frac{3.00000006}{n}.$$

Thus, if $a > 0$ or $a \leqslant -2$, then $|\Lambda| > 0.99999994/n$ and (5) cannot be satisfied. Now, the only value which remains is $a = -1$, which implies $b = 3$. However, the pair $(a, b) = (-1, 3)$ is not a solution of (2).

We conclude that in Case 2 there are no solutions of (2) in the range $3 \leqslant \Lambda \leqslant (n \log n)/7$.

Case 3. $i_0 = 3$.

In this last case we whoose $j = 2$ and $k = 1$. The estimations of the logarithms appearing in Λ are now

$$\begin{aligned} &\log \delta = -\log n - \frac{z_0}{n}, && 2 < z_0 < 4.00000008, \\ &\log \left|\frac{\xi_1}{\xi_2}\right| = \log n + \frac{z_1}{n}, && 1.99999999 < z_1 < 3.00000002, \\ &\log \left|\frac{\varepsilon_1}{\varepsilon_2}\right| = 2 \log n + \frac{z_2}{n}, && 0.99999999 < z_2 < 4. \end{aligned} \tag{10}$$

Therefore,

$$\Lambda = (-1 + a + 2b) \log n + \frac{-z_0 + az_1 + bz_2}{n}.$$

Suppose that $\Lambda \leqslant (n \log n)/7.1$. If $1-a-2b \neq 0$, then

$$|\Lambda| \geqslant \log n - \frac{4.00000008}{n} - \frac{7.00000002}{n} \Lambda \geqslant 0.014 \log n,$$

which contradicts (5) if $A \geqslant 3$. Therefore, in the range $3 \leqslant A \leqslant (n \log n)/7.1$ we must have $a = 1 - 2b$. In this case, $b \neq 0$, 1 and

$$A = \left(\log \delta + \log \left|\frac{\xi_1}{\xi_2}\right|\right) + b\left(\log \left|\frac{\varepsilon_1}{\varepsilon_2}\right| - 2 \log \left|\frac{\xi_1}{\xi_2}\right|\right).$$

By (10), the coefficient of b lies between $-5.00000005/n$ and $0.00000002/n$, and therefore it is not a priori known that it lies "far from zero." But, if, instead of the first equation of (3), we use the sharper estimate $n+1 < \xi_1 < n+1+n^{-2}$, then in the third equation of (10) we get $z_2 < 3.00000001$; this implies that the coefficient of b is $< -0.99999997/n$. Also, by (10),

$$-\frac{2.00000009}{n} < \log \delta + \log \left|\frac{\xi_1}{\xi_2}\right| < \frac{1.00000002}{n}.$$

Thus, if $b \geqslant 2$ or $b \leqslant -3$, then $|A| > 0.99999982/n$ and (5) cannot be satisfied. The values which remain are $(a, b) = (5, -2)$, $(3, -1)$ and only the second one is a solution of (2). We conclude that in Case 3 the only solution of Eq. (2) in the range $3 \leqslant A \leqslant (n \log n)/7.1$ is $(a, b) = (3, -1)$.

By combining the conclusions of the three previous cases we get the following: if $n > 10^8$, then the only solution of (2) in the range $3 \leqslant A \leqslant (n \log n)/7.1$ is $(a, b) = (3, -1)$, which corresponds to the solution $(x, y) = (1, -n)$ of the original equation $(*)$.

If $A \leqslant 2$, it is easy to check that the only solutions of (2) are

$$(a, b) = (0, 0), (1, 0), (0, 1), (-1, -1),$$

corresponding to the following solutions of $(*)$:

$$(x, y) = (1, 0), (0, -1), (1, -1), (-n-1, -1).$$

We call the above four solutions (together with $(1, -n)$) the *trivial* solutions of the equation.

Hence, *if* $n > 10^8$ *and if* $(*)$ *has any non-trivial solution, then* $A > (n \log n)/7.1$.

5. For Sufficiently Large n There Is No Non-trivial Solution

Suppose now that $n > 10^8$ and that $(*)$ has a non-trivial solution. Then, in view of the final conclusion of Section 4 and relation (4), we have

$$\frac{n \log n}{7.1} < 3.6922806 \cdot 10^{28} \cdot \log^2 n \cdot (\log \log n)^2,$$

which implies $n < 3.67 \cdot 10^{32}$.

Thus, we have proved the following result:

THEOREM. *If* $n \geqslant 3.67 \cdot 10^{32}$, *then the only solutions of the diophantine equation*

$$x^3 - nx^2y - (n+1)xy^2 - y^3 = 1$$

are

$$(x, y) = (1, 0), (0, -1), (1, -1), (-n-1, -1), (1, -n).$$

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