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The Complete Solution in Integers of $X^3 + 3Y^3 = 2^n$

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Some general remarks are made concerning the equation $f(x, y) = q^n$ in the integral unknowns x, y, n, where f is an integral form and $q > 1$ is a given integer. It is proved that the only integral triads (x, y, n) satisfying $x^3 + 3y^3 = 2^n$ are $(x, y, n) = (-1, 1, 1), (1, 1, 2), (-7, 5, 5), (5, 1, 7)$.

1. INTRODUCTION

In 1933 Mahler [4] proved that if $f(x, y)$ is a binary form with integer coefficients and such that among the linear factors in the factorization of f at least three are pairwise distinct (in the sense that their quotient is not a constant) then

$$P[f(x, y)] \to \infty \qquad \text{if } \max(|x|, |y|) \to \infty \text{ with } (x, y) = 1,$$

where for an integer a, $P[a]$ is the greatest prime factor of a.

Since then very interesting theorems of an analogous character have been discovered, as one can see in [5].

An immediate consequence of the theorem of Mahler is that if f is a binary form as above and q is a given integer >1, then the equation

$$f(x, y) = q^n \tag{1.1}$$

has at most finitely many integer solutions (x, y, n) with $(x, y) = 1$.

However, the interesting problem of finding as sharp an upper bound as possible for the number of these solutions or, still better, of finding all of them for any particular equation, has not been studied so far.

In a previous paper [6] the author attempted a first study of the diophantine equation

$$x^3 + 3y^3 = 2^n. \tag{1.2}$$

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He proved that the only solution to (1.2) with xy odd and n even is $(x, y, n) = (1, 1, 2)$. Making use of p-adic arguments, he also proved that if n is odd $\geqslant 3$, then for a given odd x_0 (resp. y_0) there is at most one solution (y, n) (resp. (x, n)) to $x_0^3 + 3y^3 = 2^n$ (resp. $x^3 + 3y_0^3 = 2^n$).

Analogous arguments can be used to other particular equations of the form (1.1) when f is a cubic form of a negative discriminant; the author has worked several (unpublished) examples.

However, the complete solution of (1.2) remained an open problem in [6]. It is the purpose of this paper to prove that the only solutions to (1.2) with xy odd are given by $(x, y, n) = (-1, 1, 1), (1, 1, 2), (-7, 5, 5), (5, 1, 7)$.

2. The Complete Solution of (1.2)

Throughout this paragraph we assume xy odd.

Lemma 1. *For a given n, (1. 2) has at most one solution in integers x, y.*

Proof. We work in $\mathbf{Q}(\xi)$, where $\xi^3 = 3$ (ξ real). The number of divisor classes is 1, $\varepsilon = -2 + \xi^2 > 0$ is a fundamental unit ([1, Table 2]) and $1, \xi, \xi^2$ is an integral basis ([2, Table, p. 141]). The factorization of the divisor (2) into prime divisors is $(2) = (-1 + \xi)(1 + \xi + \xi^2)$, where $(-1 + \xi)$ and $(1 + \xi + \xi^2)$ have degrees 1 and 2, respectively.

Write (1.2) in the form of the divisor equation

$$(x + y\xi)(x^2 + xy\xi + y^2\xi^2) = (-1 + \xi)^n (1 + \xi + \xi^2)^n.$$

Since, as it is easily seen, the divisors in the left-hand side are relatively prime, it follows that neither of them is divisible by both prime divisors of the right-hand side. This means that $(x + y\xi) = (-1 + \xi)^n$ or $(1 + \xi + \xi^2)^n$. The second instance must, however, be rejected as it is seen on taking norms. Therefore we must have

$$x + y\xi = (-2 + \xi^2)^m (-1 + \xi)^n, \qquad m \in \mathbf{Z}.$$

Consider now two different solutions of (1.2) corresponding to the same n, (x_1, y_1) and (x_2, y_2). Then,

$$x_1 + y_1\xi = (-2 + \xi^2)^p (-1 + \xi)^n, \qquad x_2 + y_2\xi = (-2 + \xi^2)^q (-1 + \xi)^n$$

and, without loss of generality, we suppose that $q > p$. Then,

$$x_2 + y_2\xi = (-2 + \xi^2)^m (x_1 + y_1\xi)$$

where $m=q-p$ is a positive integer. On equating the coefficient of ξ^2 in the right-hand side to zero, we get

$$\begin{aligned} x_1 &\left[\binom{m}{1}(-2)^{m-1}+3^2\binom{m}{4}(-2)^{m-2}+\cdots\right] \\ &+y_1\left[3\binom{m}{2}(-2)^{m-2}+3^3\binom{m}{5}(-2)^{m-5}+\cdots\right]=0. \end{aligned} \tag{2.1}$$

Let $3^r \parallel m$, $r \geqslant 0$ ($\parallel$ means "exactly divides"). Then $3^r \parallel x_1\binom{m}{1}(-2)^{m-1}$, while every

$$3^{2k}\binom{m}{3k+1}=\frac{3^{2k}m}{3k+1}\binom{m-1}{3k}, \qquad k \geqslant 1$$

and every

$$3^{2k+1}\binom{m}{3k+2}=\frac{3^{2k+1}m}{3k+2}\binom{m-1}{3k+1}, \qquad k \geqslant 0$$

is divisible by 3^{r+1}. Thus, (2.1) is impossible and this completes the proof of the lemma.

We also quote the following result ([6, Th. 1]):

LEMMA 2. *The diophantine equation $x^3+3y^3=4^n$ is impossible if xy is odd and $n>1$.*

Now we are in a position to prove the

THEOREM. *The diophantine equation* (1.2) *is impossible if $n>7$.*

Proof. Let $n>7$. From (1.2) we get $(x^3-3y^3)^2+12x^3y^3=2^{2n}$, i.e.,

$$X^2+3Y^3=2^{2(n-1)} \tag{2.2}$$

where we have put $X=(x^3-3y^3)/2$ and $Y=xy$ (obviously, X, Y are odd and relatively prime).

Since the diophantine equation $x^3+y^3=3z^3$ is impossible ([3, Ch. XXI]), n is not divisible by 3.

First let $n \equiv 1 \pmod 3$. Then, we put $2(n-1)=3r$, where r is an even number >4 and from (2.2) we get

$$(2^r)^3-3Y^3=X^2.$$

Using the notations of the proof of Lemma 1, as well as the information given there concerning $\mathbf{Q}(\xi)$, we easily get

$$2^r-Y\xi=\varepsilon^i(u+v\xi+w\xi^2)^2, \qquad u, v, w \in \mathbf{Z},\ i=0, 1.$$

If $i = 0$, then

$$2^r = u^2 + 6vw, \qquad -Y = 3w^2 + 2uv, \qquad v^2 + 2uw = 0 \tag{2.3}$$

from which we see that u is even, $(u, w) = 1$. Then, since we may suppose without loss of generality that w is positive, we have, in view of the third equation of (2.3), $u = -2a^2$, $w = b^2$, $(2a, b) = 1$, $v = 2ab$ and, on substituting in the first equation of (2.3),

$$a(a^3 + 3b^3) = 2^{r-2}.$$

From this it follows that $a = \pm 1$ and $a^3 + 3b^3 = \pm 2^{r-2}$, which is impossible by Lemma 2.

If $i = 1$, then

$$2^r = -2u^2 + 9w^2 + 6uv - 12vw, \qquad -Y = 3v^2 - 6w^2 - 4uv + 6uw,$$
$$u^2 - 4wu - 2v^2 + 6vw = 0.$$

Then, w is even, v is odd and, regarding the third as a second-degree equation in u, we conclude that its discriminant $4w^2 - 6wv + 2v^2$ must be a perfect square. This, however, is impossible, as it is seen mod 4.

Next, let $n \equiv 2 \pmod 3$. Then, we put $2(n-1) = 3r + 2$, where r is even and $\geqslant 4$. Now (2.2) becomes

$$(2^{r+1})^3 - 6Y^3 = 2X^2. \tag{2.4}$$

We work in $\mathbf{Q}(\delta)$, where $\delta^3 = 6$ (δ real). In this field the class-number is 1, $\zeta = 1 - 6\delta + 3\delta^2 > 0$ is a fundamental unit, 1, δ, δ^2 is an integral basis (see the references given in the proof of Lemma 1) and $(2) = (2 - \delta)^3$ is the factorization of (2) into prime divisors. Now, from (2.4) we get

$$2^{r+1} - Y\delta = \zeta^i(2-\delta)(u + v\delta + w\delta^2)^2, \qquad u, v, w \in \mathbf{Z},\ i = 0, 1.$$

If $i = 0$, then

$$2^r = u^2 - 3v^2 - 6uw + 12vw, \qquad -Y = -u^2 + 12w^2 + 4uv - 12vw,$$
$$v^2 - uv - 3w^2 + 2uw = 0 \tag{2.5}$$

from which uv is odd and $(u, w) = 1$. On solving the third equation of (2.5) in v, we find

$$u = (3a^2 - b^2)/2, \qquad w = (a^2 - b^2)/4, \qquad v = (u \pm ab)/2$$

where ab is odd and $(a, b) = 1$. The substitution $p = (a + b)/2$, $q = (a - b)/2$, where p, q are relatively prime and have opposite parities, gives the expressions of u, v, w in terms of p, q and on substituting these expressions in the first of (2.5), we get an impossible mod 4 relation.

Next, let $i = 1$. Then,

$$2^r = -8u^2 - 39v^2 + 216w^2 + 72uv - 78uw - 96vw \tag{2.6}$$

$$-Y = -13u^2 + 72v^2 - 96w^2 - 32uv + 144uw - 156vw \tag{2.7}$$

$$0 = 6u^2 - 8v^2 - 39w^2 - 13uv - 16uw + 72vw$$

and the last relation can be put in the form

$$(-2u + 2v + 3w)^2 = (-u + 4v - 4w)(2u + 3v - 12w). \tag{2.8}$$

Now, the determinant of the coefficients of the three linear forms appearing in (2.8) is ± 1, therefore the two factors in the right-hand side of (2.8) are relatively prime (else $(X, Y) \neq 1$) and we may also suppose that they are positive (on taking $(-u, -v, -w)$ instead of (u, v, w), if necessary). Then,

$$-u + 4v - 4w = a^2, \qquad 2u + 3v - 12w = b^2, \qquad -2u + 2v + 3w = ab$$

from which we find the values of u, v, w in terms of a, b and on substituting in (2.6) we get

$$b(b^3 - 6a^3) = 2^r$$

so that we must have $b = \pm 2^s$, $s \geqslant 1$ and $\pm 2^{3s} - 6a^3 = \pm 2^{r-s}$. On substituting a by $-a$, if necessary, we may consider only the case with the upper signs, i.e.,

$$2^{3s-1} - 3a^3 = 2^{r-s-1}. \tag{2.9}$$

Since Y is odd, we see from (2.7) that u is odd and, therefore, a is odd too. Therefore, in (2.9) we must have

$$r = s + 1, \qquad 1 + 3a^3 = 2^{3s-1}$$

where $3s - 1$ is even, since r is even. Then, by Lemma 2, $3s - 1 = 2$ and $r = 2$, which contradicts our assumption that $r \geqslant 4$.

This completes the proof of the theorem.

COROLLARY. *The only integer solutions* (x, y, n) *of* (1.2) *with* xy *odd, are* $(x, y, n) = (-1, 1, 1)$, $(1, 1, 2)$, $(-7, 5, 5)$, $(5, 1, 7)$.

Proof. This is an immediate consequence of the theorem just proved and Lemma 1.

We conclude our paper by proposing the following

PROBLEM. *Let $N(p)$ denote the number of integer solutions of $x^3 + 3y^3 = p^n$ with $(x, y) = 1$. Decide if $N(p)$ is bounded as $p \to \infty$ and, if unbounded, determine a good upper bound for $N(p)$ exhibiting its growth rate. (proposed by D. J. Lewis).*

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