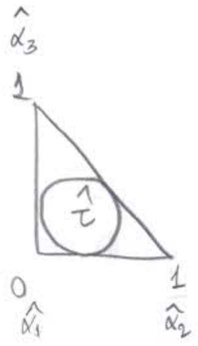
Εφμ-107 Αριθμητική Λύση ΜΔΕ

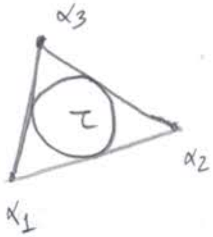
12η εξ αποστάσεως διάλεξη

19/5/2020

5μμ-7μμ

(zoom)





Αναδρομή

ΛΗΜΜΑ 1 Έστω $v \in H^m(\tau)$, $\hat{v} = v \circ F$, $m = 0, 1, 2$. Τότε

$$|\hat{v}|_{m,\hat{\tau}} \leq C \, \|A\|_2^m \, |\det(A)|^{-1/2} \, |v|_{m,\tau}$$

$$|v|_{m,\tau} \leq C \, \|A^{-1}\|_2^m \, |\det(A)|^{1/2} \, |\hat{v}|_{m,\hat{\tau}}$$

ΛΗΜΜΑ 2 $\|A\|_2 \leq \dfrac{h}{\hat{\rho}}$ $\quad \|A^{-1}\|_2 \leq \dfrac{\hat{h}}{\rho}$

$h = diam(\tau)$, $\hat{h} = diam(\hat{\tau})$

ρ η ακτίνα του μεγαλύτερου κύκλου που περιέχει στο τ

$\hat{\rho}$ η ακτίνα του μεγαλύτερου κύκλου που περιέχει στο $\hat{\tau}$

ΛΗΜΜΑ 3 (Bramble-Hilbert)

$$\inf_{\hat{p} \in P^1(\hat{\tau})} \|\hat{v} - \hat{p}\|_2 \leq C(\hat{\tau}) \, |\hat{v}|_{2,\hat{\tau}} \quad \forall \hat{v} \in H^2(\hat{\tau})$$

Τέλος αναδρομής

<u>ΛΗΜΜΑ 4</u>. Έστω $\hat{v} \in H^2(\hat{\tau})$. Then, there exists a constant $C(\hat{\tau})$ τ.ω.

$$|\hat{v} - I_{\hat{\tau}}\hat{v}|_{m,\hat{\tau}} \leq C(\hat{\tau})\, |\hat{v}|_{2,\hat{\tau}}$$

<u>Αποδ</u>. Έχουμε δείξει ότι $I_{\hat{\tau}}\hat{p} = \hat{p}$ $\forall \hat{p} \in P^1(\hat{\tau})$. Άρα:

$$\hat{v} - I_{\hat{\tau}}\hat{v} = \hat{v} - \hat{p} + \hat{p} - I_{\hat{\tau}}\hat{v} = (\hat{v} - \hat{p}) - (I_{\hat{\tau}}\hat{v} - I_{\hat{\tau}}\hat{p})$$
$$= (\hat{v} - \hat{p}) - I_{\hat{\tau}}(\hat{v} - \hat{p}) \quad \forall \hat{p} \in P^1(\hat{\tau}).$$

Επειδή: $I_{\hat{\tau}}\hat{p} = \sum_{\ell=1}^{3} \hat{p}(\hat{a}_\ell)\hat{\varphi}_\ell$ και $H^2(\hat{\tau}) \subset C(\hat{\tau})$, έχουμε:

$$\forall \hat{w} \in H^2(\hat{\tau}): \quad \|I_{\hat{\tau}}\hat{w}\|_{2,\hat{\tau}} \leq \sum_{\ell=1}^{3} |\hat{w}(\hat{a}_\ell)|\, \|\hat{\varphi}_\ell\|_{2,\hat{\tau}} \leq C(\hat{\tau})\, \|\hat{w}\|_{\infty,\hat{\tau}} \leq C(\hat{\tau})\, \|\hat{w}\|_{2,\hat{\tau}}. \qquad (I)$$

Επομένως:

$$|\hat{v} - I_{\hat{\tau}}\hat{v}|_{m,\hat{\tau}} \leq |\hat{v} - \hat{p}|_{m,\hat{\tau}} + |I_{\hat{\tau}}(\hat{v} - \hat{p})|_{m,\hat{\tau}} \leq \|\hat{v} - \hat{p}\|_{2,\hat{\tau}} + \|I_{\hat{\tau}}(\hat{v} - \hat{p})\|_{2,\hat{\tau}}$$
$$\underset{(I)}{\leq} C(\hat{\tau})\, \|\hat{v} - \hat{p}\|_{2,\hat{\tau}} \quad \forall \hat{p} \in P^1(\hat{\tau}).$$

(12.3

Έτσι:

$$|\hat{v}-I_{\hat{\tau}}\hat{v}|_{m,\hat{\tau}} \leq C(\hat{\tau}) \inf_{\hat{p}\in P^1(\hat{\tau})} \|\hat{v}-\hat{p}\|_{2,\hat{\tau}}$$

$$\overset{B-H}{\leq} C(\hat{\tau})\cdot |\hat{v}|_{2,\hat{\tau}} \qquad \boxtimes$$

Πρόταση. $|v-I_\tau v|_{m,\tau} \leq \frac{h_\tau^2}{\rho_\tau^m} |v|_{2,\tau}, \quad m=0,1$

Αποδ. Από το Λήμμα 1 έχουμε

$$|v-I_\tau v|_{m,\tau} \leqslant C \, \|A^{-1}\|_2^m \, |\det(A)|^{+1/2} \left| (v-I_\tau v)\circ F \right|_{m,\hat{\tau}}$$

$$\leqslant C \frac{h_{\hat{\tau}}^m}{\rho_\tau^m} |\det(A)|^{+1/2} |\hat{v}-I_{\hat{\tau}}\hat{v}|_{m,\hat{\tau}}$$

$$\overset{\text{Λήμμα 4}}{\leq} C(\hat{\tau}) \frac{h_{\hat{\tau}}^m}{\rho_\tau^m} |\det(A)|^{+1/2} |\hat{v}|_{2,\hat{\tau}}$$

$$\leq C(\hat{\tau}) \frac{1}{\rho_\tau^m} |\det(A)|^{+1/2} |\hat{v}|_{2,\hat{\tau}}$$

$$\overset{\text{Λήμμα 1}}{\leq} C(\hat{\tau})\,\frac{1}{\rho_\tau^{m}}\,|\cancel{\det(A)}|^{+1/2}\,C\cdot\|A\|^{2}\,|\cancel{\det(A)}|^{-1/2}\,|v|_{2,\tau}$$

$$\overset{\text{Λήμμα 2}}{\leq} C(\hat{\tau})\,\frac{1}{\rho_\tau^{m}}\cdot\frac{h_\tau^{2}}{\rho_{\hat{\tau}}^{2}}\,|v|_{2,\tau}$$

$$\leq C(\hat{\tau})\cdot\frac{h_\tau^{2}}{\rho_\tau^{m}}\,|v|_{2,\tau} \qquad \blacksquare$$

Global estimate

ΠΡΟΤΑΣΗ: Έστω $I: H^2(\Omega) \to S^1$, $Iv = \sum_{\alpha\in K(\Omega)} v(\alpha)\,\varphi_\alpha$, όπου $B=\{\varphi_\alpha \in S : \alpha\in K\}$ η Lagrange βάση του S^1

κορυφές τριγώνων

Υποθέτοντας ότι $\frac{h_\tau}{\rho_\tau} \leq \sigma$ έπεται:

$$\|v - Iv\|_{m,\Omega} \leq C\, h^{m}\, |v|_{2,\Omega} \qquad \forall v\in H^2(\Omega)$$

$m=0,1$

$h = \max_{\tau\in T} h_\tau$

(12.5

Αποδειξη:

$$\|v-Iv\|_{m,\Omega} \leq c\left\{\sum_{\tau\in T}\|v-Iv\|^2_{m,\tau}\right\}^{1/2} = \left\{\sum_{\tau\in T}\|v-I_\tau v\|^2_{m,\tau}\right\}^{1/2}$$

$$\leq C\left\{\sum_{\tau\in T}\left(\frac{h_\tau^2}{\rho_\tau^m}\right)^2 |v|^2_{2,\tau}\right\}^{1/2}$$

$m=0$ $$\|v-Iv\|_{0,\Omega} \leq c h^2 \left\{\sum_{\tau\in T}|v|^2_{2,\tau}\right\}^{1/2} \leq c h^2 |v|_{2,\Omega}$$

$m=1$ $$\|v-Iv\|_{1,\Omega} \leq c\left\{\sum_{\tau\in T}\frac{h_\tau^4}{\rho_\tau^2}|v|^2_{2,\tau}\right\}^{1/2} \leq c\sigma\left\{\sum_{\tau\in T} h_\tau^2 |v|^2_{2,\tau}\right\}^{1/2}$$

$$\leq c\, h\, |v|_{2,\tau}$$

12.6

$$\mathcal{B}(\hat{u}_D,\varphi) = \int_\Omega f\varphi\,dx \quad \forall \varphi \in \tilde{V}_D = \{p \in C(\bar\Omega) : p|_\tau \in P^1(\tau)\ \forall p \in T\}$$

$p|_{\partial\Omega} = 0$

σταν ∂Ω πολυγωνικό

$$\mathcal{B}(u_D,\varphi) = \int_\Omega f\varphi\,dx \quad \forall \varphi \in \tilde{V}_D$$

$$\mathcal{B}(v,w) = (A\nabla v,\nabla w) + (q v,w) \quad \forall v,w \in H^1(\Omega)$$

$e = \hat{u}_D - u_D$ τότε:

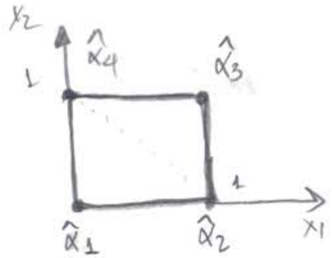
$$\mathcal{B}(e,\varphi) = 0 \quad \forall \varphi \in \tilde{V}_D$$

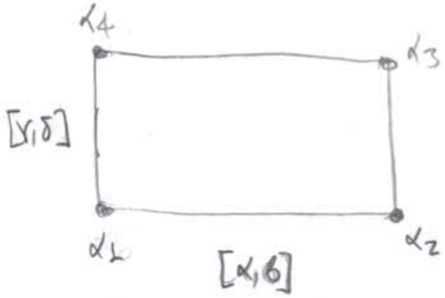
$$\Rightarrow \mathcal{B}(e,e) = \mathcal{B}(e,\hat{u}_D - u_D) = \mathcal{B}(e,\varphi - u_D) \quad \forall \varphi \in \tilde{V}_D$$

$$\Rightarrow c_E \|e\|_1^2 \le C\|e\|_1 \|\varphi - u_D\|_1 \ \forall \varphi \in \tilde{V}_D$$

$$\Rightarrow \|e\|_1 \le c \inf_{\varphi \in \tilde{V}_D} \|\varphi - u_D\|_1 \le c\,\|u_D - \underbrace{I u_D}_{\in \tilde{V}_D}\|_1 \le c\,h \cdot |u_D|_{2,\Omega}$$

Nitche $\Rightarrow$ $\|e\| \le c\,h^2 |u_D|_{2,\Omega}$





$\hat{\alpha}_1=(0,0)$, $\hat{\alpha}_2=(1,0)$,
$\hat{\alpha}_3=(1,1)$, $\hat{\alpha}_4=(0,1)$

$F\ \hat{q} \to q$.

$F(x)=Ax+b$ με $F(\hat{\alpha}_j)=\alpha_j$, $j=1,2,4$. $\qquad b=F(0)=F(\hat{\alpha}_1)$

$$x\in q \Rightarrow x=\alpha_1+\lambda_1(\alpha_2-\alpha_1)+\lambda_2(\alpha_4-\alpha_1)$$
$$=(1-\lambda_1-\lambda_2)\,\alpha_1+\lambda_1\alpha_2+\lambda_2\alpha_4, \qquad \forall \lambda_1,\lambda_2\in[0,1].$$

$$F(\hat{x})= A\left[(1-\lambda_1-\lambda_2)\hat{\alpha}_1+\lambda_1\hat{\alpha}_2+\lambda_2\hat{\alpha}_4\right]+b$$
$$=(1-\lambda_1-\lambda_2)A\hat{\alpha}_1+\lambda_1 A\hat{\alpha}_2+\lambda_2 A\hat{\alpha}_4+b$$
$$=(1-\lambda_1-\lambda_2)F(\hat{\alpha}_1)+\lambda_1F(\hat{\alpha}_2)+\lambda_2F(\hat{\alpha}_4)$$
$$=(1-\lambda_1-\lambda_2)\,\alpha_1+\lambda_1\alpha_2+\lambda_2\alpha_4 \in q, \quad \forall \hat{x}\in\hat{q}.$$

Έτσι η F είναι 1-1 και επί από το $\hat{q}$ στο q. Επιπλέον: $F(\hat{\alpha}_3)=A(\hat{\alpha}_2+\hat{\alpha}_4)+b=$

$$=(A\hat{\alpha}_2+b)+(A\hat{\alpha}_4+b)-b=\alpha_2+\alpha_4-\alpha_1=(\alpha_2-\alpha_1)+(\alpha_4-\alpha_1)+\alpha_1=\alpha_3.$$

12.8

Η βάση Lagrange του $Q^1(\hat{q})$ είναι:

$$\hat{B} = \operatorname{span}\{ \underbrace{(1-\hat{x}_1)\hat{x}_2}_{\hat{q}_4}, \underbrace{(1-\hat{x}_1)(1-\hat{x}_2)}_{\hat{q}_1}, \underbrace{\hat{x}_1(1-\hat{x}_2)}_{\hat{q}_2}, \underbrace{\hat{x}_1\hat{x}_2}_{\hat{q}_3}\}$$

Ξέραμε ότι $\dim(Q^1(\hat{q}))=4$ και επειδή τα $(\hat{q}_i)_{i=1}^{4}$ είναι γραμμικά ανεξάρτητα
αποτελούν βάση. 12.1