Αριθμητική Λύση ΜΔΕ

6η εξ' αποστάσεως διάλεξη

Παρασκευή 24/4/2020

(5μμ-7μμ)

(Zoom meeting)

Version 1

Αναδρομή

ΠΡΟΒΛΗΜΑ

$u_t = u_{xx} + f(u) \quad \forall x \in (\alpha, \beta), \forall t \in (0,T]$

$u(t,\alpha) = u(t,\beta) = 0 \quad \forall t \in (0,T]$

$u(0,x) = v(x) \quad \forall x \in [\alpha,\beta]$

$v(\alpha) = v(\beta) = 0$

$f \in C^1(\mathbb{R};\mathbb{R})$ με $\sup_{\mathbb{R}} |f'| < +\infty$

ΜΕΘΟΔΟΣ CRANK-NICOLSON / ΠΕΠΕΡΑΣΜΕΝΩΝ ΣΤΟΙΧΕΙΩΝ

ΒΗΜΑ 1º: $U^0 \in H$ προσέγγιση του v

ΒΗΜΑ 2º: Για $m = 0, \dots, N-1$, βρές $U^{m+1} \in H$ τ.ω.

$$\left(\frac{U^{m+1} - U^m}{\Delta t_{m+1}}, \varphi\right) + B\left(\frac{U^{m+1} + U^m}{2}, \varphi\right) = 0 \quad \forall \varphi \in H$$

$\longmapsto \|v - U^0\| \le C\, \Delta x^2$

$$\max_{0 \le m \le N} \|U^m - u^m\| \le C\,(\Delta t^2 + \Delta x^2)$$

↳ εξαρτάται από την u όχι όμως από τις διαμερίσεις

Μη γραμμική περίπτωση:

↱ χώρος πεπερασμένων στοιχείων

ΒΗΜΑ 1: $U^0 \in H$ προσέγγιση της v

ΒΗΜΑ 2: Για $m = 0, \dots, N-1$, βρές $U^{m+1/2} \in H$ τ.ω.

$$\left(\frac{U^{m+1/2} - U^m}{\Delta t_{m+1/2}}, \varphi\right) + B\left(\frac{U^{m+1/2} + U^m}{2}, \varphi\right) = \left(f(U^m), \varphi\right) \quad \forall \varphi \in H$$

$\Delta t_{m+1/2} := \frac{\Delta t_{m+1}}{2}$

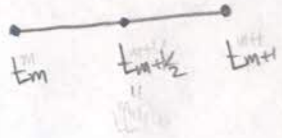
και $U^{m+1} \in H$ τ.ω

$$\left(\frac{U^{m+1} - U^m}{\Delta t_{m+1}}, \varphi\right) + B\left(\frac{U^{m+1} + U^m}{2}, \varphi\right) = \left(f(U^{m+1/2}), \varphi\right) \quad \forall \varphi \in H.$$

(6.2

Σφάλμα συνέπειας

$$\frac{u^{m+1/2}-u^{m}}{\Delta t_{m+1/2}} - \partial_x^2\left(\frac{u^{m+1/2}+u^{m}}{2}\right) = f(u^m) + R_1^m$$



$$\frac{u^{m+1}-u^{m}}{\Delta t_{m+1}} - \partial_x^2\left(\frac{u^{m+1}+u^{m}}{2}\right) = f(u^{m+1/2}) + R_2^m$$

όπου:

$$R_1^m(x) = \int_{t_m}^{t_{m+1/2}} f'(u(s,x))\,u_t(s,x)\,ds + \frac{1}{2}\left[\int_{t_{m+1/4}}^{t_{m+1/2}} (t_{m+1/2}-s)\,u_{ttt}(s,x)\,ds + \int_{t_m}^{t_{m+1/4}} (s-t_m)\,u_{ttt}(s,x)\,ds\right]$$

$$-\frac{1}{2\Delta t_{m+1/2}}\left[\int_{t_{m+1/4}}^{t_{m+1/2}} (t_{m+1/2}-s)^2\,u_{ttt}(s,x)\,ds + \int_{t_m}^{t_{m+1/4}} (t_m-s)^2\,u_{ttt}(s,x)\,ds\right] = O(\Delta t_{m+1})$$

$$R_2^m(x) = \frac{1}{2}\left[\int_{t_{m+1/2}}^{t_{m+1}} (t_{m+1}-s)\,u_{ttt}(s,x)\,ds + \int_{t_m}^{t_{m+1/2}} (s-t_m)\,u_{ttt}(s,x)\,ds\right] - \frac{1}{2\Delta t_{m+1}}\left[\int_{t_{m+1/2}}^{t_{m+1}} (t_{m+1}-s)^2\,u_{ttt}(s,x)\,ds + \int_{t_m}^{t_{m+1/2}} (t_m-s)^2\,u_{ttt}(s,x)\,ds\right]$$

$$+\frac{1}{2}\left[\int_{t_{m+1/2}}^{t_{m+1}} (t_{m+1}-s)\,(f(u))_{ttt}(s,x)\,ds + \int_{t_m}^{t_{m+1/2}} (s-t_m)\,(f(u))_{ttt}(s,x)\,ds\right] = O(\Delta t_{m+1})$$

Εκτίμηση σφάλματος

$\theta^m = Eu^m - U^m$ για $m=0,\dots,N$

$\theta^{m+1/2} = Eu^{m+1/2} - u^{m+1/2}$ για $m=0,\dots,N-1$.

$$\left(\frac{\theta^{m+1/2}-\theta^m}{\Delta t_{m+1/2}},\varphi\right) + B\left(\frac{\theta^{m+1/2}+\theta^m}{2},\varphi\right) = \left[\left(E\left(\frac{u^{m+1/2}-u^m}{\Delta t_{m+1/2}}\right),\varphi\right) + B\left(E\left(\frac{u^{m+1/2}+u^m}{2}\right),\varphi\right)\right]$$

$$- \left[\left(\frac{U^{m+1/2}-U^m}{\Delta t_{m+1/2}},\varphi\right) + B\left(\frac{U^{m+1/2}+U^m}{2},\varphi\right)\right]$$

$$= \left(E\left(\frac{u^{m+1/2}-u^m}{\Delta t_{m+1/2}}\right),\varphi\right) + B\left(\frac{u^{m+1/2}+u^m}{2},\varphi\right) - \left(f(U^m),\varphi\right)$$

$$= \underbrace{\left(E\left(\frac{u^{m+1/2}-u^m}{\Delta t_{m+1/2}}\right) - \frac{u^{m+1/2}-u^m}{\Delta t_{m+1/2}},\varphi\right)}_{A_1^m} + \underbrace{\left(f(u^m),\varphi\right) - f(U^m),\varphi)}_{B_1^m} + \underbrace{(R_1^m,\varphi)}_{r_1^m}$$

Ανάλογα:

$$\left(\frac{\vartheta^{m+1}-\vartheta^m}{\Delta t_{m+1}},\varphi\right)+\beta\left(\frac{\vartheta^{m+1}+\vartheta^m}{2},\varphi\right)=\underbrace{\left(E\left(\frac{u^{m+1}-u^m}{\Delta t_{m+1}}\right)-\left(\frac{u^{m+1}-u^m}{\Delta t_{m+1}}\right),\varphi\right)}_{A_2^m}+\underbrace{\left(f(u^{m+1/2})-f(U^{m+1/2}),\varphi\right)}_{B_2^m}+(R_2^m,\varphi).$$

B_1^m, B_2^m:

$$\begin{aligned}\|f(u^{m+1/2})-f(U^{m+1/2})\| &\le \sup_{\mathbb{R}}|f'|\;\|u^{m+1/2}-U^{m+1/2}\|\\ &\le \sup_{\mathbb{R}}|f'|\left(\|u^{m+1/2}-Eu^{m+1/2}\|+\|Eu^{m+1/2}-U^{m+1/2}\|\right)\\ &\le \sup_{\mathbb{R}}|f'|\left(C\,\Delta x^2\,\|\partial_x^2u^{m+1/2}\|+\|\vartheta^{m+1/2}\|\right).\end{aligned}$$

ανάλογα:

$$\|f(u^m)-f(U^m)\|\le\sup_{\mathbb{R}}|f'|\left(C\,\Delta x^2\|\partial_x^2u^m\|+\|\vartheta^m\|\right).$$

Εκτίμηση A_1^m, A_2^m : $\left\| E\left(\frac{u^{m+1/2}-u^m}{\Delta t_{m+1/2}}\right) - \frac{u^{m+1/2}-u^m}{\Delta t_{m+1/2}} \right\| \leq C\, \Delta x^2 (\Delta t_{m+1/2})^{-1} \int_{t_m}^{t_{m+1/2}} \|\partial_x^2 \partial_t u(s,\cdot)\|\, ds$

$$\left\| E\left(\frac{u^{m+1}-u^m}{\Delta t_{m+1}}\right) - \left(\frac{u^{m+1}-u^m}{\Delta t_{m+1}}\right) \right\| \leq C\, \Delta x^2\, (\Delta t_{m+1})^{-1} \int_{t_m}^{t_{m+1}} \|\partial_x^2 \partial_t u(s,\cdot)\|\, ds$$

Εκτίμηση R_1^m, R_2^m : $\|R_1^m\| \leq C\, \Delta t_{m+1/2}$

$$\|R_2^m\| \leq C\, (\Delta t_{m+1})^2$$

$\varphi = \vartheta^{m+1/2} + \vartheta^{m}$ $\qquad$ $\varphi = \vartheta^{m+1} + \vartheta^{m}$

$$\|\vartheta^{m+1/2}\|^2 - \|\vartheta^m\|^2 + \frac{\Delta t_{m+1/2}}{2} |\vartheta^{m+1/2} + \vartheta^m|_1^2 \leq \Delta t_{m+1/2} \left(\|A_1^m\| + \|B_1^m\| + \|R_1^m\| \right) \left(\|\vartheta^{m+1/2}\| + \|\vartheta^m\| \right)$$

$$\Rightarrow \quad \|\vartheta^{m+1/2}\| - \|\vartheta^m\| \leq C\, \Delta t_{m+1} \left(\Delta x^2 \max_{0\leq t\leq T} \|\partial_x^2 u\| + \|\vartheta^m\| \right) + C \Delta x^2 \int_{t_m}^{t_{m+1/2}} \|\partial_t \partial_x^2 u(s,\cdot)\| \, ds + C \Delta t_{m+1} \|R_1^m\|$$

Ανάλογα:

$$\|\vartheta^{m+1}\| - \|\vartheta^m\| \leq C\, \Delta t_{m+1} \left(\Delta x^2 \max_t \|\partial_x^2 u\| + \|\vartheta^{m+1/2}\| \right) + C \Delta x^2 \int_{t_m}^{t_{m+1/2}} \|\partial_t \partial_x^2 u(s,\cdot)\| \, ds + C\Delta t_{m+1} \|R_2^m\|$$

$$\leq C\, \Delta t_{m+1} \Delta x^2 \cdot \max_t \|\partial_x^2 u\| + C \Delta x^2 \int_{t_m}^{t_{m+1/2}} \|\partial_t \partial_x^2 u(s,\cdot)\| \, ds + C \Delta t_{m+1} \|R_2^m\|$$

$$+ C \Delta t_{m+1} \left(\|\vartheta^m\| + C \Delta t_{m+1} \left(\Delta x^2 \max_t \|\partial_x^2 u\| + \|\vartheta^m\| \right) + C \Delta x^2 \int_{t_m}^{t_{m+1/2}} \|\partial_t \partial_x^2 u(s,\cdot)\| \, ds + C \Delta t_{m+1} \|R_1^m\| \right)$$

$$\leq \underbrace{b_m}_{O(\Delta t_{m+1} \Delta x^2 + \Delta t_{m+1} \Delta t_{m+1}^2)} + C \Delta t_{m+1} \|\vartheta^m\| \quad , m = 0, \dots, N-1$$

Διακριτό Gronwall $\Rightarrow$ $\max \|\vartheta^n\| \sim C\, (\Delta x^2 + \Delta t^2)$, εφόσον $\|\vartheta^0\| \leq C \Delta x^2$ πχ. όταν $U^0 = I_{\Delta x} v$ ή $E v$ ή $P v$.

Επιπλέον έχουμε:

$$\|\theta^{n+\frac{1}{2}}\| \leq \|\theta^n\|(1+C\Delta t_{n+1}) + C\Delta t_{n+1}\Delta x^2 \max_{[0,T]}\|u_{xx}\|$$

$$+ C\Delta x^2 \int_{t_n}^{t_{n+1/2}} \|u_{*xx}\|\, ds$$

$$+ C\Delta t_{n+1}\, O\left(\Delta t_{n+1}/2\right)$$

$$\leq O(\Delta x^2 + \Delta t^2), \quad n = 0, \ldots, N-1$$

Επομένως με υπολογιστικό κόστος 2N γραμμικών συστημάτων
προσαρμόσαμε 2N χρονικούς κόμβους, και επιτυγχάνουμε
υπολογίζουμε
τάξη 2 στο χρόνο και στο χώρο.